Lectures on Quantum Field Theory

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QFT Redhead QFT

Lecture I

Relativistic Wave Equations

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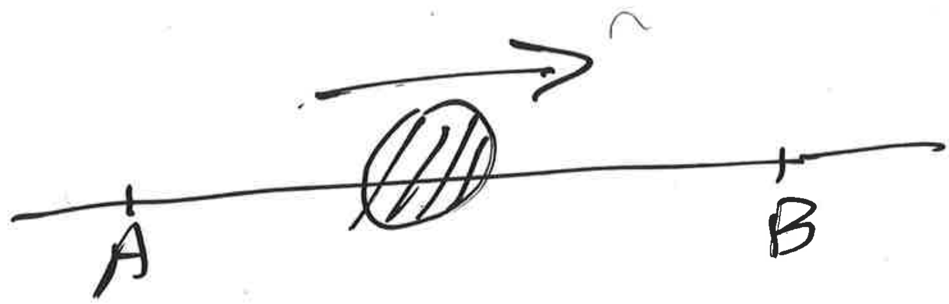
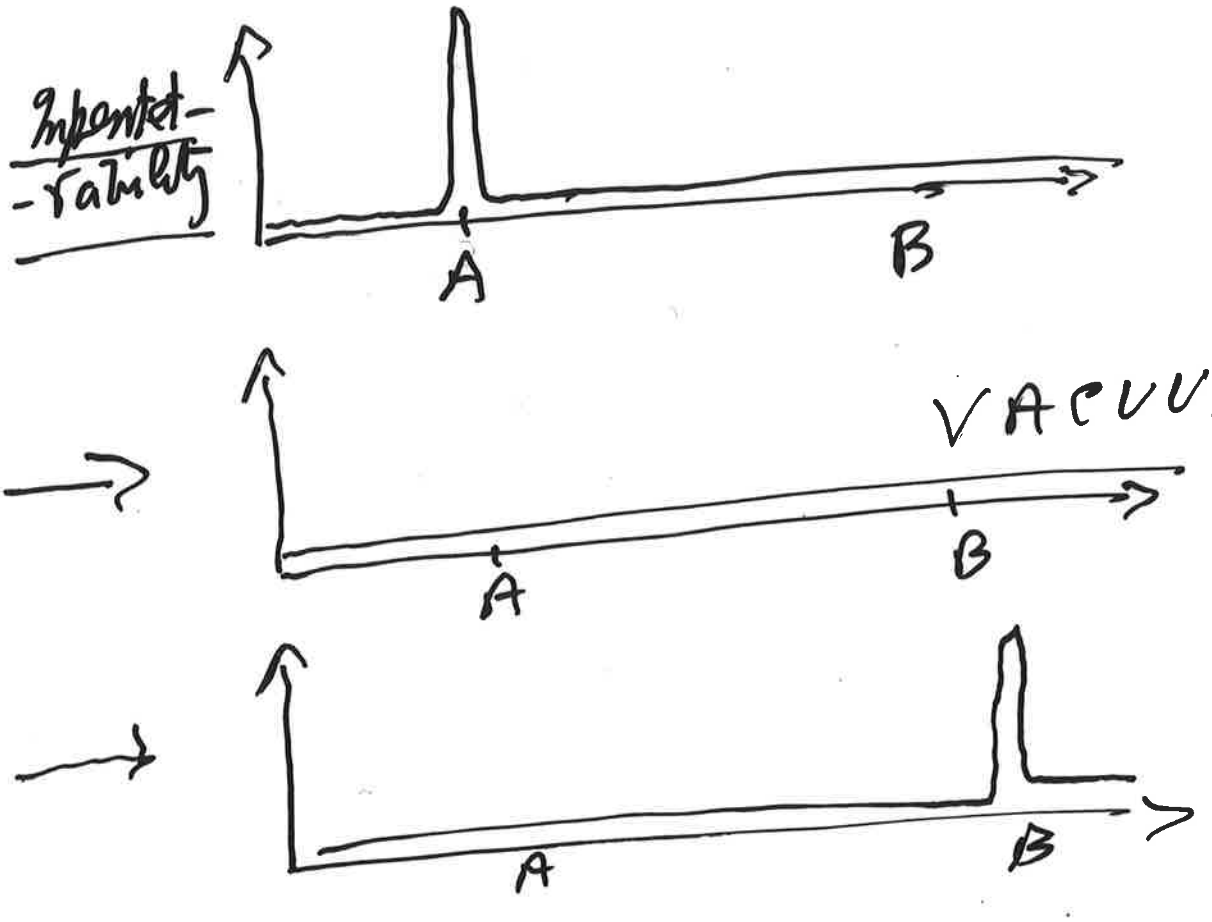
Particles and Fields

Particle motion from A to B



Field Description in terms of Impenetrability



RELATIVISTIC WAVE EQUATIONS

$$E^2 = p^2c^2 + m^2c^4$$

Klein - Gordon

$$E \to i\hbar\frac{\partial}{\partial t}\ , \quad p \to -i\hbar\nabla$$

So we get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\psi = 0$$

where $\mu = mc/\hbar$

Dirac

$$E = (\underline{\alpha}\cdot\underline{p})c + \beta m$$

So $$i\hbar\frac{\partial\psi}{\partial t} = -i\underline{\alpha}\cdot\underline{\nabla}\cdot c\,\psi + \beta mc^2\psi$$

where β & components of α anti commute.

(6)

(2)

NR localization

'Eigenstate' of position (non-normalizable) at the point $\underset{\sim}{\xi}$ is (in 3-dimensions)

$$\phi^{\xi}(\underset{\sim}{x}) = \delta^3(\underset{\sim}{x} - \underset{\sim}{\xi}) \qquad (1)$$

Inner product of ϕ^{ξ} and $\phi^{\xi'}$ is

$$\langle \phi^{\xi'} | \phi^{\xi} \rangle = \int d^3\underset{\sim}{x}\ \phi^{\xi' *}(\underset{\sim}{x})\, \phi(\underset{\sim}{x})\, d^3\underset{\sim}{x}$$

$$= \delta(\xi - \xi') \qquad (2)$$

So for $\xi \neq \xi'$, ϕ^{ξ}, $\phi^{\xi'}$ are orthogonal

Note $\|\phi^{\xi}\| = \sqrt{\delta(0)}$ is infinite

To deal with this we introduce wave-packet $\psi^{\xi}(\underset{\sim}{x}) = \int F_{\underset{\sim}{\xi}}(\underset{\sim}{\xi}')\, \phi^{\xi'}(\underset{\sim}{x})\, d^3\xi'$

centred around $\underset{\sim}{\xi}$

So $\psi^{\xi}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d^3\underset{\sim}{k}\ \tilde{F}_{\xi}(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$

where $\tilde{F}_{\xi}(\underset{\sim}{k})$ is F.T. of $F_{\xi}(\xi')$

Then $\|\psi^{\xi}(x)\|^2 = \int d^3\underset{\sim}{k}\ |\tilde{F}_{\xi}(\underset{\sim}{k})|^2 d^3\underset{\sim}{k}$

Note more generally that, if $F(\underset{\sim}{k})$ is F.T. of $\psi(\underset{\sim}{x})$

then $\langle \psi_1 | \psi_2 \rangle = \int d^3\underset{\sim}{k}\, \overline{F_1(\underset{\sim}{k})} \cdot F_2(\underset{\sim}{k})$ — (3)

Relativistic Localization

Inner product (3) is <u>not</u> invariant

Soln: If

$$\psi(\underset{\sim}{x}, t) = (2\pi)^{-3/2} \int \frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega(\underset{\sim}{k})t)} \cdot F(\underset{\sim}{k})$$

where $\omega(\underset{\sim}{k}) \underset{Df}{=} \sqrt{m^2 + \underset{\sim}{k}^2}$

Then note that if ψ is a scalar field (K.G. Eqn), then since $\frac{d^3\underset{\sim}{k}}{\omega(\underset{\sim}{k})}$ is <u>invariant</u> measure on the mass-shell $F(\underset{\sim}{k})$ is ~~an~~ <u>scalar</u> function of $\underset{\sim}{k}$, i.e. $F'(\underset{\sim}{k}') = F(\underset{\sim}{k})$ where $\underset{\sim}{k}'$ is the boost of $\underset{\sim}{k}$ <u>on the mass-shell</u>

Then invariant inner product is given by.

$$\langle \psi_1 | \psi_2 \rangle = \int \frac{d^3 \underset{\sim}{k}}{\omega(\underset{\sim}{k})} F_1(\underset{\sim}{k})^* \cdot F_2(\underset{\sim}{k}) \quad \text{--- (4)}$$

This replaces (3) above.

The usual rules for finding amplitudes and probabilities then apply with the <u>new</u> inner product.

Let us work at $t = 0$,

So $$\psi(x, 0) = (2\pi)^{-3/2} \int \frac{d^3 \underset{\sim}{k}}{\omega(\underset{\sim}{k})} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \cdot F(\underset{\sim}{k}) \quad \text{-- (5)}$$

Try various choices for $F(\underset{\sim}{k})$:

(1) $F(\underset{\sim}{k}) = \omega(\underset{\sim}{k})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\xi}} \cdot (2\pi)^{-3/2}$

$$\Rightarrow \psi(x, 0) \to \frac{1}{(2\pi)^3} \int d^3 \underset{\sim}{k}\, e^{i\underset{\sim}{k}(\underset{\sim}{x} - \underset{\sim}{\xi})} = \delta(\underset{\sim}{x} - \underset{\sim}{\xi})$$

This is the <u>same</u> as $\phi^{\xi}(\underset{\sim}{x})$ for the N.R. case.

<u>But</u> now $$\langle \phi^{\xi'} | \phi^{\xi} \rangle = (2\pi)^{-3} \int d^3_{\sim}k\, \omega(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{\xi}' - \underset{\sim}{\xi})} \neq \delta^3(\underset{\sim}{\xi} - \underset{\sim}{\xi}')$$

So try again!

$$(2) \quad F(\underset{\sim}{k}) = \sqrt{\omega(\underset{\sim}{k})}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\xi}'} \cdot (2\pi)^{-3/2}$$

yields

$$\phi^{\xi}(\underset{\sim}{x}) = (2\pi)^{-3} \int \frac{d^3\underset{\sim}{k}}{\sqrt{\omega(\underset{\sim}{k})}}\, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{\xi})} \qquad (6)$$

$$\left(\neq \delta(\underset{\sim}{x} - \underset{\sim}{\xi})\right)$$

But now

$$\langle \phi^{\xi'} | \phi^{\xi} \rangle = (2\pi)^{-3} \int \frac{d^3k}{\omega(\underset{\sim}{k})} \sqrt{\omega(k)} \cdot \sqrt{\omega(\underset{\sim}{k})}\, e^{ik\cdot(\xi'-\xi)}$$

$$= \delta(\xi - \xi')$$

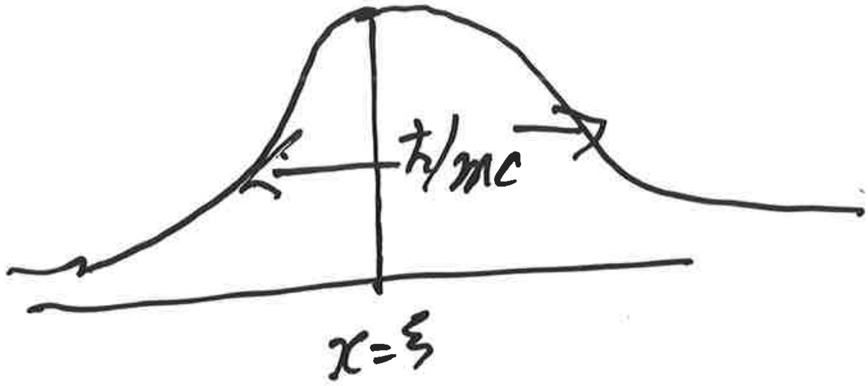
as we want!

The state (6) is the famous <u>Newton-Wigner</u> state

In fact $\phi^{\xi}(\underset{\sim}{x}) \sim K_{5/4}(|\underset{\sim}{x} - \xi|)$

$$\sim e^{-|x-\xi|}$$

where K_n is <u>modified Hankel function of the first kind</u>



$|\underline{x}-\underline{\xi}|$

Note: spread of ϕ^{ξ} in $\underset{\sim}{x}$-space has nothing to do with interactions or pair production etc

– we are just 'doing' 1-particle R Q T

N.W. Wave-packets

$$\psi^{\xi}(\underset{\sim}{x}) = \int F_{\xi}(\underset{\sim}{\xi}') \, \phi^{\xi'}(\underset{\sim}{x}) \, d^3\xi'$$

where $F_{\xi}(\underset{\sim}{\xi}') = \langle \phi^{\underset{\sim}{\xi}'} | \psi^{\underset{\sim}{\xi}} \rangle$

and as in N-R analysis we can write

$$\psi^{\xi}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int \frac{d^3\underset{\sim}{k}}{\sqrt{\omega(\underset{\sim}{k})}} \, \tilde{F}_{\xi}(\underset{\sim}{k}) \, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

where $\tilde{F}_{\xi}(\underset{\sim}{k})$ is F.T. of $F_{\underset{\sim}{\xi}}(\underset{\sim}{\xi}')$

For general state $\psi(\underset{\sim}{x})$ what is probability amplitude $G(\underset{\sim}{\xi})$ for finding particle at $\underset{\sim}{\xi}$?

Expand $\psi(\underset{\sim}{x}) = \int G(\underset{\sim}{\xi})\,\phi^{\xi}(\underset{\sim}{x})\,d^3\underset{\sim}{\xi}$

Then it is easy to show that

$$G(\xi) = (2\pi)^{-3/2}\int \frac{d^3\underset{\sim}{k}}{\sqrt{\omega(\underset{\sim}{k})}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{\xi}}\, A(\underset{\sim}{k}) \quad \ldots (7)$$

$$\left(\text{where } \psi(x) = (2\pi)^{-3/2}\int \frac{d^3k}{\omega(\underset{\sim}{k})}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, A(\underset{\sim}{k})\right)$$

$$= (2\pi)^{-3}\int d^3\underset{\sim}{k}\int d^3\underset{\sim}{x}\,\sqrt{\omega(\underset{\sim}{k})}\; e^{i\underset{\sim}{k}\cdot(\underset{\sim}{\xi}-\underset{\sim}{x})}\,\psi(\underset{\sim}{x}). \quad \ldots (8)$$

Then Probability Density of finding particle at ξ $= |G(\underset{\sim}{\xi})|^2$.

We shall apply (7) & (8) to 3 problems

(12) (18)

A How to Boost $\psi^0(\underset{\sim}{x})$?

In this case $A(\underset{\sim}{k}) = \hat{F}_0(\underset{\sim}{k})\sqrt{\omega(\underset{\sim}{k})}$

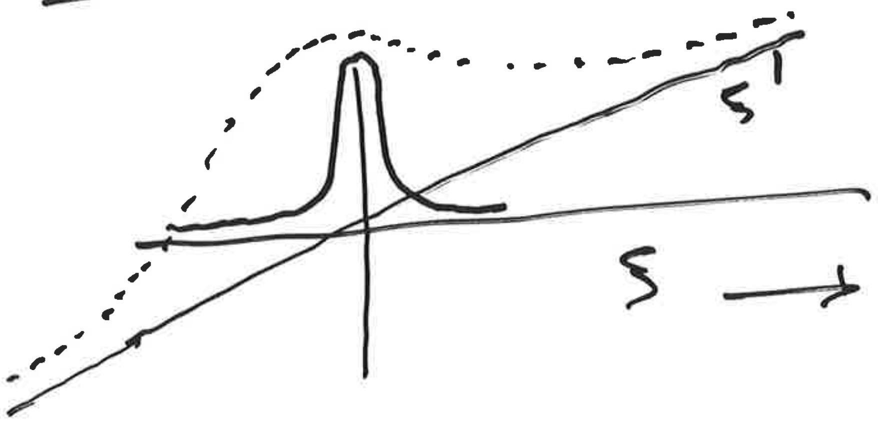
So transformed state in momentum space is

$$A'(\underset{\sim}{k}) = \hat{F}_0'(\underset{\sim}{k})\sqrt{\omega'(\underset{\sim}{k})}$$

Hence

$$G(\xi') = (2\pi)^{-3/2}\int d^3\underset{\sim}{k}\,\sqrt{\frac{\omega'(\underset{\sim}{k})}{\omega(\underset{\sim}{k})}}\cdot \hat{F}_0'(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{\xi}'}$$

$$\neq F_0(\xi') \;!!$$



B Time - Evolution of $\psi^0(\underline{x})$

At time t $\psi^0(\underline{x})$ becomes

$$(2\pi)^{-3/2}\int \frac{d^3\underline{k}}{\sqrt{\omega(\underline{k})}}\,\tilde{F}_0(\underline{k})\,e^{i(\underline{k}\cdot\underline{x}-\omega(\underline{k})t)}$$

So at time t

$$G(\underline{\xi}) = (2\pi)^{-3/2}\int d^3\underline{k}\;\tilde{F}_0(\underline{k})\,e^{i(\underline{k}\cdot\underline{\xi}-\omega(\underline{k})t)}$$

This is essentially the integral considered by Hegerfeldt.

$G(\underline{\xi})$ cannot vanish even for large $\underline{\xi}$, due to branch points in $\omega(\underline{k})$ at $|\underline{k}| = \pm i m$.

(14) (8)

A How to boost $\psi^0(\underset{\sim}{x})$?

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Hence

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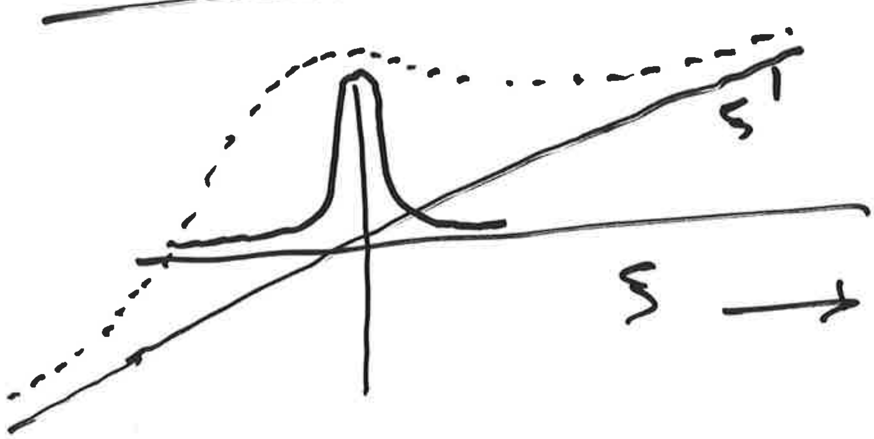
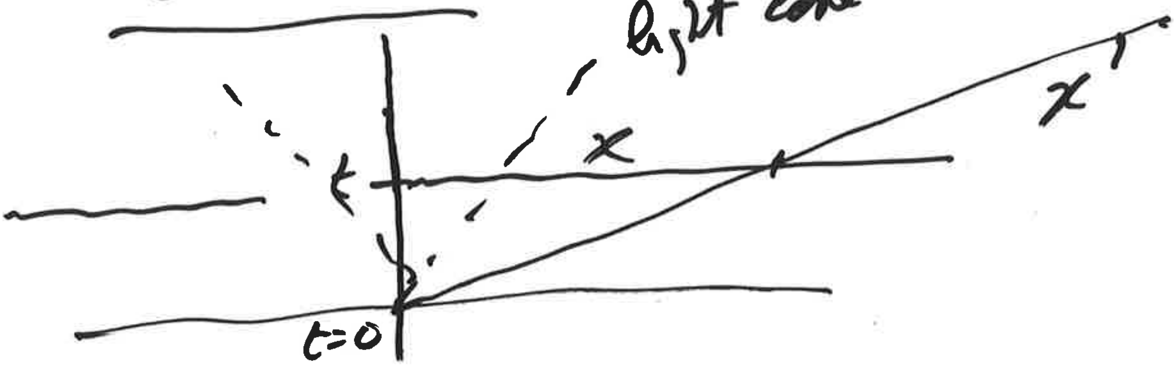


C. How does time-evolution relate to non-invariance of boost?



We have $\psi(x,t) = \psi'(x',0)$
$$= \psi'(x'(x,t),0)$$

Hence at time t,

$$G(\underset{\sim}{\xi}) = (2\pi)^{-9/2} \int d^3\underset{\sim}{k} \int d^3\underset{\sim}{k}' \int d^3\underset{\sim}{x}\; e^{i\underset{\sim}{k}'\cdot\underset{\sim}{x}'(x,t)}\, \hat{F}_0'(\underset{\sim}{k}')\, \frac{\sqrt{\omega'(k')}\cdot\sqrt{\omega(k)}}{\omega(\underset{\sim}{k}')} \cdot e^{i\underset{\sim}{k}\cdot(\underset{\sim}{\xi}-\underset{\sim}{x})}$$

N.B. for t very small the x-integral can be extended inside the light-cone with a contribution of order t^2.

(16) (11)

For a Newton-Wigner state for small t in 2-dimensional spacetime

$$G(\xi) \propto \delta(\xi) + t \cdot e^{-|\xi|} \cdot f(\xi)$$

---

where $f(\xi)$ is a complicated function of ξ

Redhead QFT

Lecture II

Second Quantization and Quantum Field Theory

De Broglie waves and the Klein – Gordon Equation

$$\psi \propto e^{-i\omega t + i\underset{\sim}{k}\cdot\underset{\sim}{x}}.$$

where $\omega = \sqrt{1+k^2}$

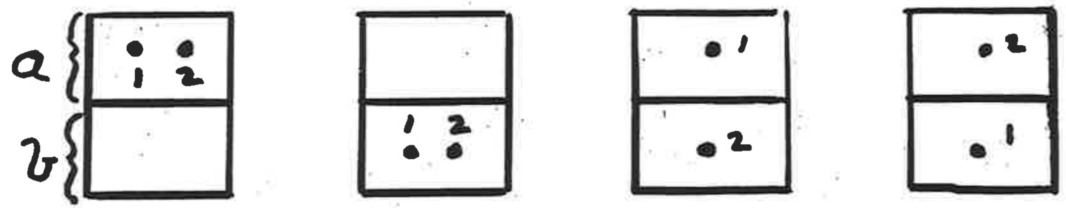
phase velocity $= \dfrac{\omega}{k} = \dfrac{\sqrt{1+k^2}}{k} > 1$

(remember 1 corresponds to the velocity of light in vacuo)

But
group velocity $= \dfrac{d\omega}{dk} = \dfrac{k}{\omega} < 1$

$\dfrac{k}{\omega} = \dfrac{\text{momentum}}{\text{energy}} =$ velocity of the particle

Statistical weights for 2- particle System



(I) (II) (III) (IV)

In Quantum Stat. Mech. (III) and (IV) are regarded as one and the same state for the purposes of assigning Statistical weights

Qantum Statistical Mechanics

Consider the 4 product wave functions

$$\psi_a(\underline{r}_1)\cdot\psi_a(\underline{r}_2) \qquad \text{I}$$

$$\psi_b(\underline{r}_1)\cdot\psi_b(\underline{r}_2 \qquad \text{II}$$

$$\psi_a(\underline{r}_1)\cdot\psi_b(\underline{r}_2) \qquad \text{III}$$

$$\psi_a(\underline{r}_2)\cdot\psi_b(\underline{r}_1) \qquad \text{IV}$$

4-dimensional vector space equally well spanned by

Symmetric
$$\left\{\begin{array}{ll} \psi_a(\underline{r}_1)\cdot\psi_a(\underline{r}_2) & \text{V} \\ \psi_b(\underline{r}_1)\cdot\psi_b(\underline{r}_2) & \text{VI} \\ \frac{1}{\sqrt{2}}\left(\psi_a(\underline{r}_1)\cdot\psi_b(\underline{r}_2)+\psi_a(\underline{r}_2)\cdot\psi_b(\underline{r}_1)\right) & \text{VII} \end{array}\right.$$

Anti symmetric $\frac{1}{\sqrt{2}}\left(\psi_a(\underline{r}_1)\cdot\psi_b(\underline{r}_2)-\psi_a(\underline{r}_2)\cdot\psi_b(\underline{r}_1)\right)$ VIII

THE INDISTINGUISHABILITY PRINCIPLE (IP)

Two particles are indistinguishable if $\langle P\phi | Q | P\phi \rangle = \langle \phi | Q | \phi \rangle$

$$\forall\, Q,\, P,\, \phi$$

IP can be taken as a restriction on <u>observables</u> $\Rightarrow$ P commutes with Q, i.e. Q is a symmetric function of particle labels $\longrightarrow$ <u>parastatistics</u>

<u>or</u> IP can be regarded as a restriction on <u>states</u> $\Rightarrow$ $P|\phi\rangle = \pm|\phi\rangle$

So Bosons and Fermions only allowed

Identity of Indiscernibles

$$\forall F\,(F(x) \longleftrightarrow F(y)) \rightarrow x = y.$$

Query? What F's should be included under the scope of the 2^{nd} order quantifier to force identity?

More generally, can the *particular* be reduced to the *universal*?

SECOND QUANTIZATION

Start with N-particle Wave Eq. for an assembly of Bosons. State is specified by giving no. n_i of particles in i^{th} 1-particle state $|u_i\rangle$ (with energy E_i)

Then $\underline{E = \sum_i n_i E_i}$

Compare with assembly of harmonic oscillators

$$* \quad \underline{E = \sum_i \left(n_i + \tfrac{1}{2}\right) E_i} \quad , \text{ if } \omega_i = E_i/\hbar$$

But * is what we would get by subjecting the $\underline{\text{1-particle W. Eq.}}$ to a $\underline{\text{second}}$ quantization
(such as K.G. Eq.)

But 2nd quantization is more <u>general</u> than the N-Particle Schrödinger Eq. because of the constraint

$$\sum_i n_i = N$$

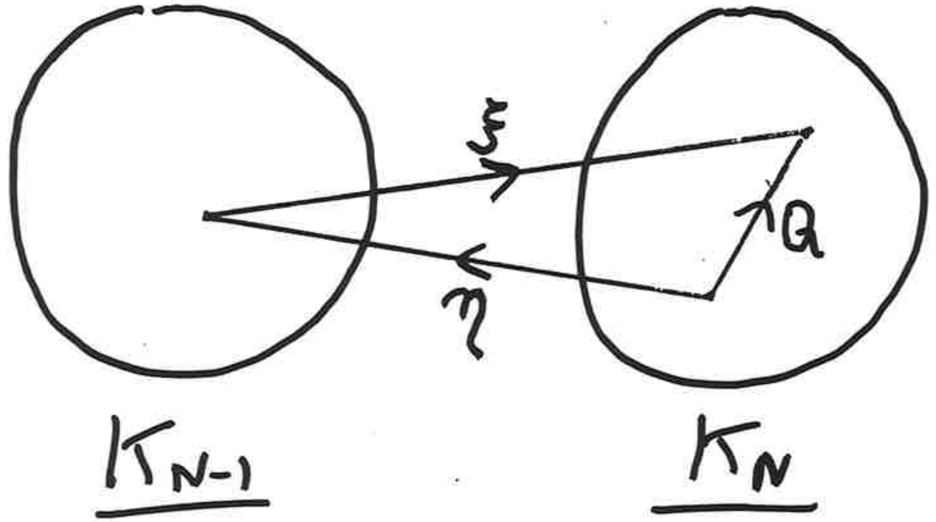
<u>FOCK SPACE</u>

$$\mathcal{F} = \underset{\downarrow \text{vacuum}}{K_0} \oplus K_1 \oplus \cdots K_N \oplus \cdots$$

<u>CREATION and ANNIHILATION OPERATORS</u>

$$\left.\begin{aligned} a_i\,|n_i\rangle &= \sqrt{n_i}\cdot|n_i-1\rangle \\ a_i^{+}\,|n_i\rangle &= \sqrt{n_i+1}\cdot|n_i+1\rangle \end{aligned}\right\}$$

Schematically we factorise
$Q = \xi \cdot \eta$



But $Q' = \xi + \eta$ would create and annihilate particles

FIELD QUANTIZATION

Real Klein - Gordon field

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)\psi = 0$$

Energy spectrum

$$E = \sum_{\underset{\sim}{k}} n_k (\hbar \omega_k) + \text{const.}$$

where $\text{const.} = \frac{1}{2}\sum_{\underset{\sim}{k}} (\hbar \omega_k)$

and $\omega_k = c\sqrt{\mu^2 + \underset{\sim}{k}^2}$

n_k are integral eigenvalues of the operator $N_k = a_k^+ a_k$

where $a_k \,|n_k\rangle = \sqrt{n_k}\,|n_k - 1\rangle$

and $a_k^+ \,|n_k\rangle = \sqrt{n_{A+1}}\,|n_k + 1\rangle$

<u>So</u> Basic Result of QFT is :

The number of particles (quanta) present with momentum $(\hbar \underline{k})$ and energy $(\hbar \omega_{\underline{k}})$ in the <u>Particle Representation</u> is just the <u>excitation number</u> $n_{\underline{k}}$ of the $\underline{k}$ - mode

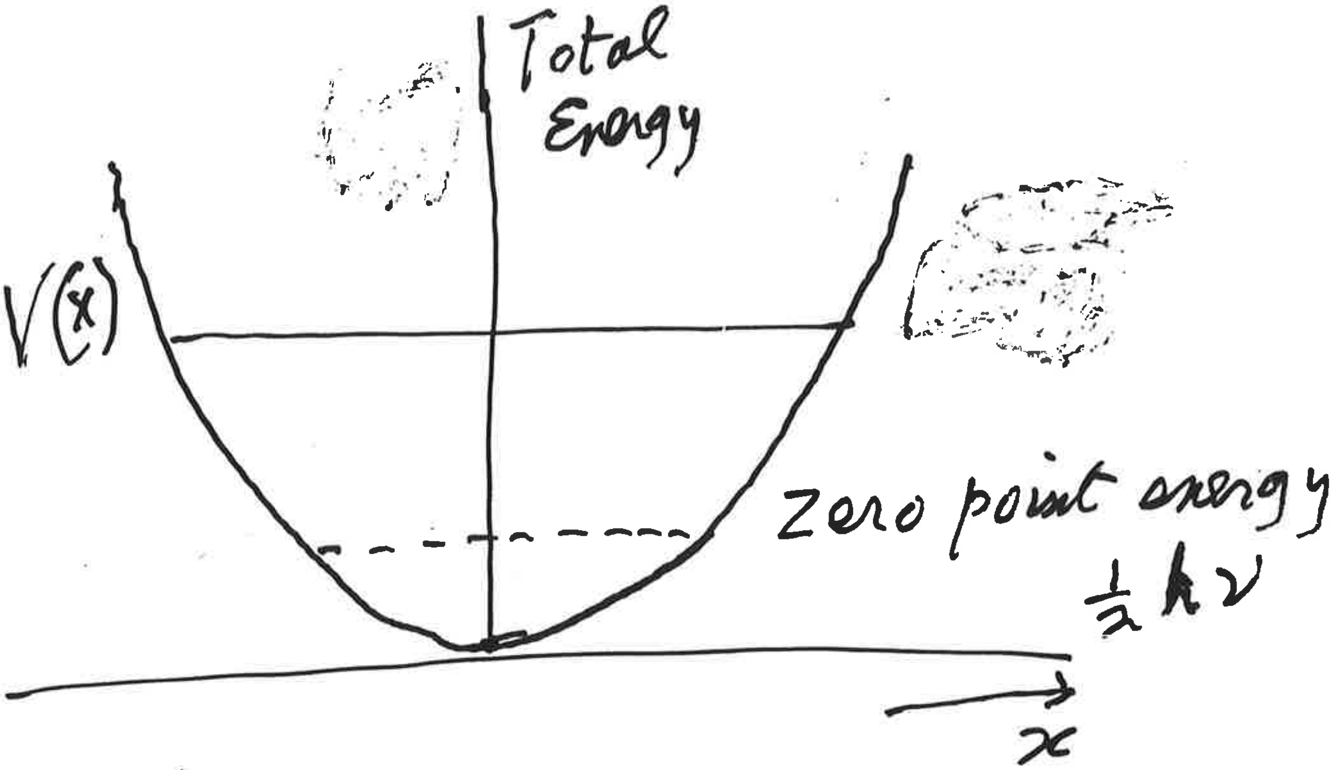
THE VACUUM

This is the state for which all the $n_{\underline{k}}$ are zero.
It is the lowest energy state of the field.
But the energy is not zero, since the field amplitude and other local quantities exhibit

Vacuum fluctuations.

The non-vanishing energy of the vacuum is called the zero-point energy of the field.

The Harmonic Oscillator in Quantum Mechanics



ν = frequency of oscillator

Heisenberg Uncertainty Principle

$\Delta p \, \Delta x \sim h$ ← Planck's Constant

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FIELD QUANTIZATION contd

Schrödinger matter field $\psi(\underset{\sim}{x})$ satisfies

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

Derive from Lagrangian density

$$L = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V(\underset{\sim}{x})\psi^*\psi$$

Canonically conjugate field is

$$\Pi(\underset{\sim}{x}) = i\hbar\psi^*$$

Impose quantization

$$[\psi(\underset{\sim}{x}), \Pi(\underset{\sim}{x}')] = i\hbar\,\delta^3(\underset{\sim}{x}-\underset{\sim}{x}')$$

Expand $\psi = \sum a_k u_k(\underset{\sim}{x})\, e^{-iE_k t/\hbar}$

$\psi^+ = \sum a_k^+ u_k^*(\underset{\sim}{x})\, e^{iE_k t/\hbar}$.

u_k are energy eigenstates of Schrödinger field

$\Rightarrow \quad [a_k, a_\ell^+] = \delta_{k\ell}$

Hamiltonian density is

$$H = \Pi\dot{\psi} - L = -\frac{i\hbar}{2m}\nabla\Pi\cdot\nabla\psi - \frac{i}{\hbar}V\Pi\psi$$

Total Hamiltonian $\mathcal{H} = \int H\, d^3\underset{\sim}{x} = \sum a_k^+ a_k E_k$

$n_k = a_k^+ a_k$

Eigenvalues of N_k

$$N_k\,|n_k\rangle = n_k\,|n_k\rangle$$

What are possible values for n_k?

Note $N_k\left(a_k|n_k\rangle\right) = (n_k - 1)\,a_k\,|n_k\rangle$

So $a_k\,|n_k\rangle = c_n\,|n_k - 1\rangle$

But $\langle n_k|N_k|n_k\rangle = \|\,a_k|n_k\rangle\|^2 \geq 0$

So N_k cannot have a negative eigenvalue, moreover $c_n = \sqrt{n}$

$\Rightarrow$ n_k is any integer ≥ 0.

$$(a_k)^{n_k}\,|n_k\rangle = \sqrt{n_k!}\;|0_k\rangle$$

Similarly $a_k^{+}\,|n_k\rangle = \sqrt{n_k+1}\;|n_k+1\rangle$

and $|n_k\rangle = \dfrac{1}{\sqrt{n_k!}}\,(a_k^{+})^{n_k}\,|0_k\rangle$

For relativistic fields

$$H = \sum\left(N_k + \tfrac{1}{2}\right)E_k$$

zero point energy. ~~zero point energy~~

Eigenvalues of $N_k = a_k^+ a_k$ with anticommutation Relations

Note $N_k^2 = a_k^+ a_k a_k^+ a_k$

$= a_k^+ a_k (1 - a_k a_k^+)$

$= N_k - a_k^+ a_k^2 a_k^+$

$= N_k$, since $a_k^2 = 0$

So eigenvalues n_k obey

$n_k^2 = n_k$,

i.e. $n_k(n_k - 1) = 0$ $\therefore$ $n_k = 0$ or 1

Pauli Exclusion Principle

Summary

The Non-relativistic Schrödinger Equation

$$\psi = V^{-\frac{1}{2}} \sum_{k} a_k e^{-i\omega_k t} e^{i k \cdot x}$$

$$\omega_k = k^2/2m$$

$$H = \sum_k \omega_k a_k^{\dagger} a_k = \sum_k \omega_k . N_k$$

where $N_k = a_k^{\dagger} a_k$.
has eigenvalues $n_k = 0, 1 2 3 \ldots$

$$Q = e \sum_k N_k.$$

Klein - Gordon Equation (complex)

$$\psi = V^{-\frac{1}{2}} \sum_k \left[a_k e^{-i\omega_k t} e^{i k \cdot x} + b_k^{\dagger} e^{i\omega_k t} e^{-i k \cdot x} \ldots \right.$$

$$\omega_k = +\sqrt{1+k^2}$$

$$H = \frac{1}{2} \sum_k \omega_k \left(a_k a_k^{\dagger} + a_k^{\dagger} a_k + b_k^{\dagger} b_k + b_k b_k^{\dagger} \right)$$

$$= \sum \omega_k \left(N_k^{+} + N_k^{-} \right) + \sum_k \omega_k$$

where $N_k^{+} = a_k^{\dagger} a_k$, $N_k^{-} = b_k^{\dagger} b_k$.

$$Q = e \sum_k \left(N_k^{+} - N_k^{-} \right)$$

Dirac Equation

$$\psi = \sum_{k,s}\Big[a_k u(k,s)\, e^{-i\omega_k t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} + b_k^{\dagger} v(k,s)\, e^{i\omega_k t}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\Big]$$

$$H = \sum_k \omega_k \left(a_k^{\dagger} a_k - b_k b_k^{\dagger}\right)$$

$$Q = e\sum_k \left(a_k^{\dagger} a_k + b_k b_k^{\dagger}\right)$$

Using anticommutation relations

$$H \to \sum_k \omega_k \left(N_k^{+} + N_k^{-}\right) - \sum_k \omega_k .$$

$$Q \to e\sum_k \left(N_k^{+} - N_k^{-}\right) + e\sum_k 1 .$$

Now redefine

$$H \to \sum_k \omega_k \left(N_k^{+} + N_k^{-}\right)$$

$$Q \to e\sum_k \left(N_k^{+} - N_k^{-}\right)$$

Query? What would commutation relations look like for the Dirac field

Would we have $[b_k, b_k^{\dagger}] = 1$ (Weinberg)

or $[b_k, b_k^{\dagger}] = -1$ (Everyone else!)

Three Routes to the Quantum Field

Fock space. $\longrightarrow$ **Quantum Field**

2nd Quantization $\longrightarrow$

Field Quantization $\longrightarrow$

$$\psi = V^{-\frac{1}{2}} \sum_{k} \hat{a}_{k} \, e^{i(\omega_k t - \underset{\sim}{k} \cdot \underset{\sim}{x})}$$

where $\hat{a}_k$ is an operator

Inequivalent Representations

Consider the case of fermions: The basis states in terms of occupation numbers can be represented as a sequence mapping a denumerable set of 1-particle states into occupation numbers 0 or 1. We can think of this sequence as an infinite binary fraction, such as 0.0101101110 .....

Fock space is the space spanned by all terminating binary fractions. This is a denumerable set and hence gives rise to a separable Hilbert space.

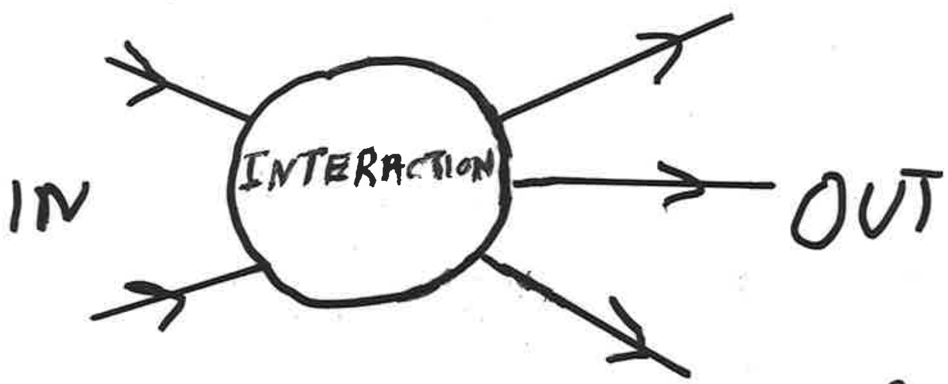
But we can also consider a space spanned by all binary fractions, including nonterminating ones. This is a nonseparable space which supports inequivalent representations of anticommutation algebra.

Redhead QFT

Lecture III

Feynman Diagrams and Virtual Particles

SCATTERING THEORY



The transition amplitude from the initial IN state to a final OUT state defines the S-Matrix

$$\langle OUT | S | IN \rangle$$

(2)

If $|\psi(t)\rangle = \sum_n c_n(t)|\phi_n\rangle$

where $|\phi_n\rangle$ are eigenstates of H_0, and

$$|\psi(-\infty)\rangle = |IN\rangle$$

Then the transition amplitude to the OUT state $|\phi_n\rangle$ is given by $c_n(\infty)$.

But compare Haag's Theorem: $\psi(t)$ and ϕ_n belong to inequivalent representations! So above expansion is mathematical nonsense ($\psi(t)$ for $t > -\infty$ is orthogonal to all the ϕ_n). But we will press on!

Integral equation for Green's function

Schrödinger equation $\left(i\hbar\frac{\partial}{\partial t}-H_0\right)\psi = 0$

Solved by

$$K_0 = \left(i\hbar\frac{\partial}{\partial t}-H_0\right)^{-1}$$

$$\left(\text{So } \left(i\hbar\frac{\partial}{\partial t}-H_0\right)K_0 = 1\right)$$

For inhomogeneous equation

$$\left(i\hbar\frac{\partial}{\partial t}-H_0\right)\psi = F$$

Soln is $\psi = K_0 F$ — but also solves homogeneous eqn with inhomog. bounding conditions $\psi(2) = K_0(2,1)\psi$

with interaction $H = H_0 + V$

Green's function is $K = \left(i\hbar\frac{\partial}{\partial t}-H\right)^{-1}$

using $A^{-1} = B^{-1} + B^{-1}(B-A)A^{-1}$

$$\Rightarrow \quad K = K_0 + K_0 V K$$ Integral Equation.

Expand

$$\psi(2) = \sum_n c_n \phi_n(x_2)\, e^{-iE_n(t_2-t_1)}$$

But for $t_2 = t_1$

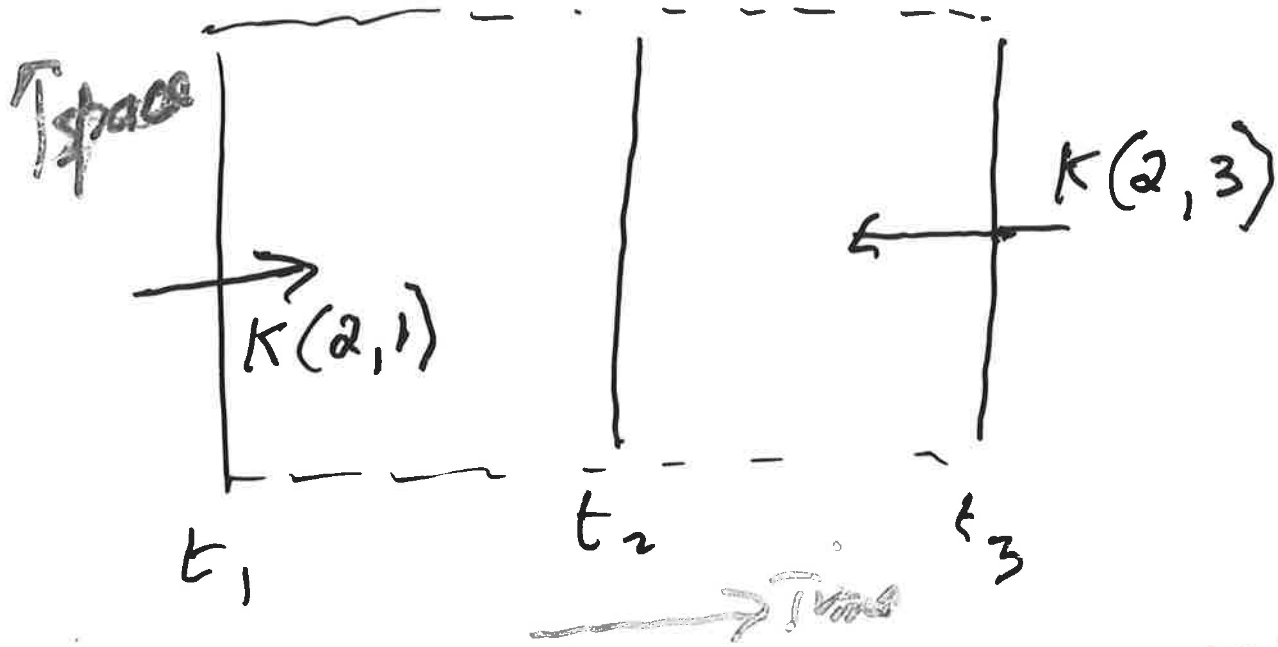
$$\psi(2) \rightarrow \psi(1) = \sum_n c_n \phi_n(x_1)$$

so $c_n = \int \psi(1)\, \phi_n^*(x_1)\, dx_1$

Hence

$$\psi(2) = \int \left[\sum_n \phi_n^*(x_1)\, \phi_n(x_2)\, e^{-iE_n(t_2-t_1)} \right] \psi(1)\, dx_1$$

But [ ] is <u>not</u> a Green's function since it satisfies homogeneous S. Eq.



For Green's theorem to apply we need a closed 4-dimensional surface ~~s~~ such as spatial slices at t_1 and t_3, and time-like boundary at spatial infinity

We define K so that $K(2,3)=0$ – it cannot propagate backwards in time

6

27

$$\langle 2 | K_0 | 1 \rangle \underset{Df}{=} K_0(2,1)$$

$$= \sum_n \phi_n^*(x_1)\, \phi_n(x_2)\, e^{-i/\hbar\, E_n (t_2 - t_1)} \cdot \theta(t_2 - t_1)$$

where $\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

FEYNMAN DIAGRAMS

$$K = K_0 + K_0 V K$$

or $$(1 - K_0 V) K = K_0$$

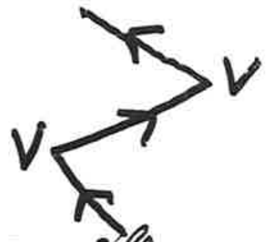
So $$K = (1 - K_0 V)^{-1} K_0$$

$$= \sum_{n=0}^{\infty} (K_0 V)^n . K_0$$

$$= \underline{K_0} + \underline{K_0 V K_0} + \underline{K_0 V K_0 V K_0} \cdots$$

No scattering

Single scattering



Double scattering

FEYNMAN PROPAGATOR

$$K(2,1) = \sum_{E_n>0,\, E_n<0} \phi_n^*(1)\, \phi_n(2)\, e^{-i/\hbar E_n (t_2 - t_1)} \times \theta(t_2 - t_1)$$

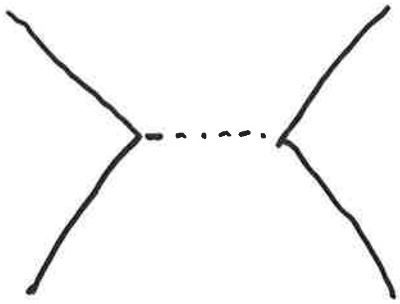
$$- \sum_{E_n<0} \phi_n^*(1)\, \phi_n(2)\, e^{-i/\hbar E_n (t_2 - t_1)}$$

$$= \begin{cases} \sum_{E_n>0} \cdots & \text{for } t_2 > t_1 \\ \sum_{E_n<0} \cdots & \text{for } t_2 < t_1 \end{cases}$$

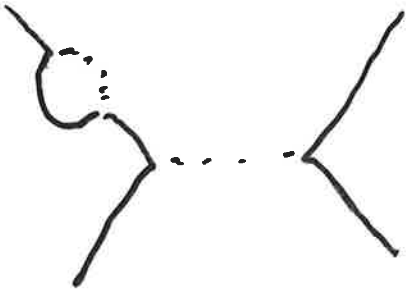
EXAMPLES of FEYNMAN DIAGRAMS

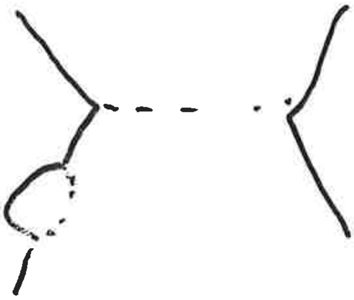
Electron - electron scattering

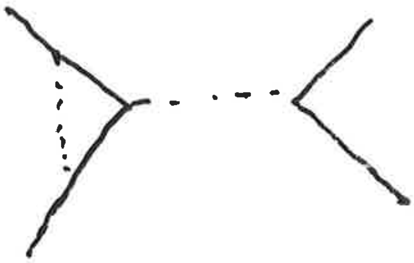
2nd order

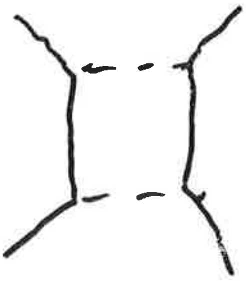


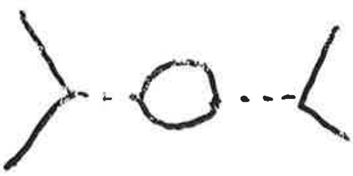
4th order.



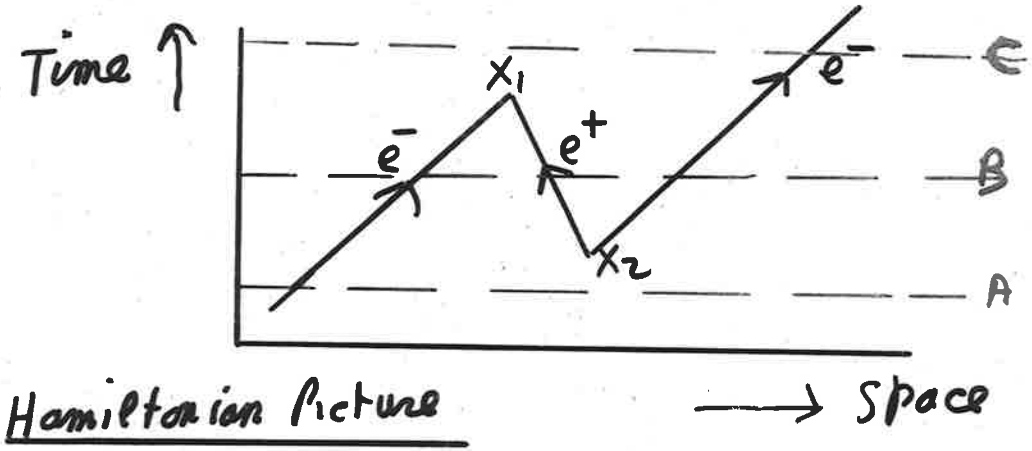




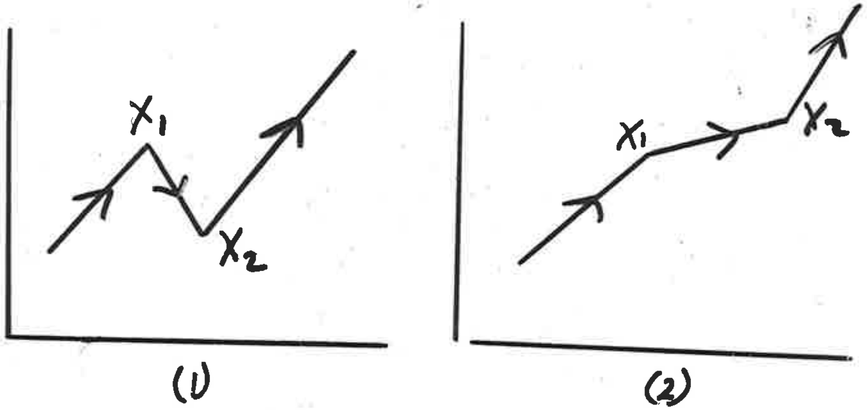




Feynman Diagrams

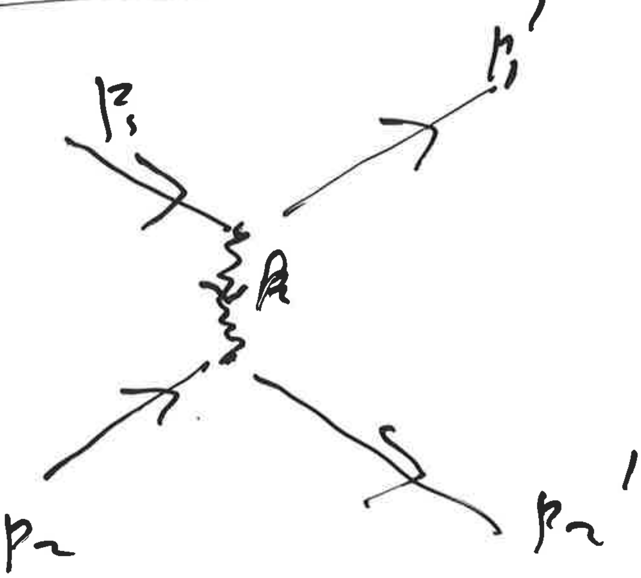


Hamiltonian Picture



Feynman Picture

VIRTUAL PARTICLES AND CONSERVATION LAWS



4-vectors
p_1, p_1', p_2, p_2', k

Conservation at each vertex gives

$$k = p_1 - p_1' = p_2' - p_2$$

$$\therefore \quad p_1' + p_2' = p_1 + p_2$$

overall conservation

But $k^2 = (p_1 - p_1')^2 = p_1^2 + p_1'^2 - 2\, p_1 \cdot p_1'$

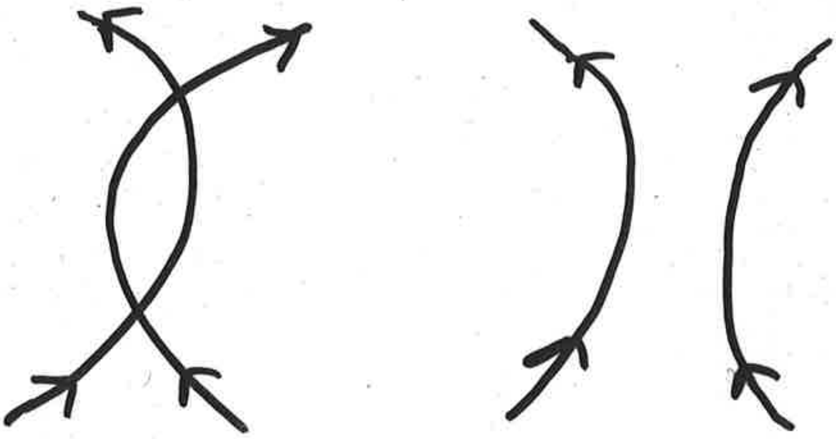
$= 2m^2 - 2(E^2 - p^2 \cos\theta)$ in c.m. frame

$= -2p^2(1 - \cos\theta)$ $\quad\theta$ = scattering angle

$= -4p^2 \sin^2 \tfrac{1}{2}\theta$

But for a real photon $k^2 = 0$!!

Does Exchange of Virtual Particles always produce Repulsion?

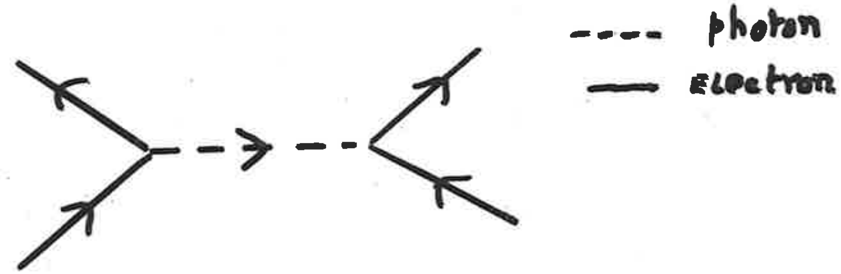


Attraction

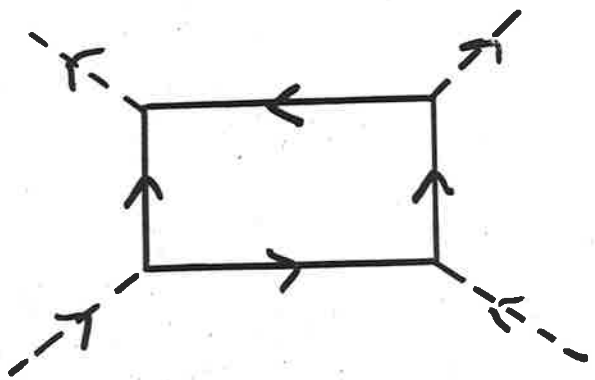
Repulsion

MATTER AND FORCE

Compare



with



So which is the 'matter' particle and which is the 'force' particle?

Classical Renormalization

Eq. of motion for an electron of radius r_0 is :—

$$m\,\ddot{\underset{\sim}{r}} = \underset{\sim}{K}^{(0)} + \underset{\sim}{K}^{(1)} + \cdots$$

where $\underset{\sim}{K}^{(0)} = -\beta \cdot \frac{e^2}{r_0 c^2}\,\ddot{\underset{\sim}{r}}$, $\beta \sim 1$

$$\underset{\sim}{K}^{(1)} = 2/3\; e^2/c^3 \cdot \dddot{\underset{\sim}{r}} \quad \text{etc}$$

We rewrite this as

$$m'\,\ddot{\underset{\sim}{r}} = \underset{\sim}{K}^{(1)} + O(r_0)$$

where $m' = m + \beta \cdot \frac{e^2}{r_0 c^2}$

We identify m' with experimental mass and <u>then</u> let $r_0 \to 0$.

Three Views on Renormalization

1. Cutoffs

2. Real infinities approach

3. Mask of Ignorance.

Redhead QFT

Lecture IV

Vacuum Fluctuations

WHAT IS THE VACUUM?

Remove all the particles, electrons, photons etc in the Universe, and you would be left with the vacuum.

Queries

1.) are you left with nothing, e.g. no space or time?

2.) Does it make sense to talk of emptying spacetime of gravitation? — cp vacuum solns of Einstein field eqs. in GR

3.) Could the notion of the vacuum depend on the state of motion of the observer? — cp Unruh Effect

We shall confine ourselves to special Relativity, but already there are many surprises!

MORE ADO ABOUT NOTHING

~~(Redhead — Found. Phys. (1999) 123)~~

Theorem 1. Any local event that can happen in some arbitrary state of a field can also happen in the vacuum.

(Hellwig & Kraus (1970))

Theorem 2 In the vacuum any measurement located at x is maximally correlated with some simultaneous measurement located at y, however far apart x and y may be.

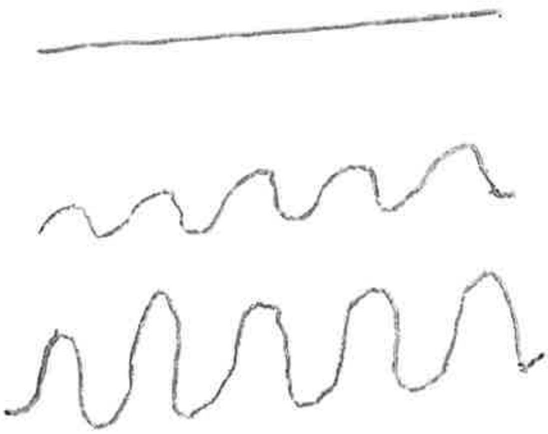
Theorem 3 Every local measurement is infinitely ambiguous, i.e. leaves infinitely many questions unanswered

Quantum Field Theory

Nonrelativistic case : Quantizing the Schrödinger field

Particles = quantized excitations of the field

Vacuum

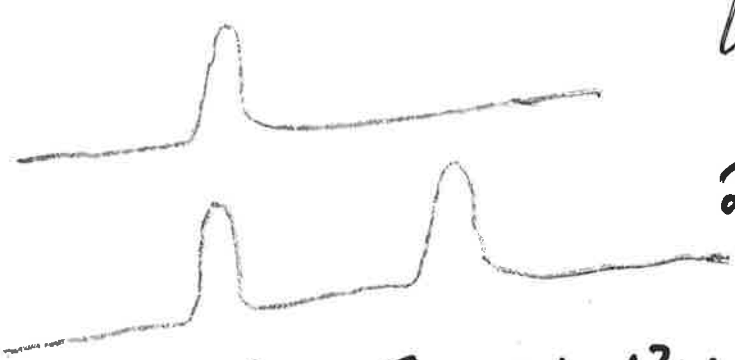


no excitation

1 particle of definite momentum

2 particles

Localized Excitations



1 particle

2 particles

$N = \int n(x) d^3x$ $\quad N_V = \int_V n(x) d^3x$

$[N_V, N_{V'}] = 0$ for disjoint $V \,\&\, V'$

$\therefore N = 0 \Rightarrow N_V = 0$ for any subvolume V.

So global vacuum
$\Rightarrow$ local vacuum

In relativistic QFT this is not true
$[N_V, N_{V'}] \neq 0$ for disjoint V, V'.

So $N = 0 \not\Rightarrow N_V = 0$.

Two reactions to this:

(1) Virtual Particles:
exist for times $\sim \hbar/mc^2$
i.e. travel at most a distance $\sim \hbar/mc$

globally vacuum $\Rightarrow$ no <u>real</u> particles
<u>But</u> locally lots of virtual particles

<u>objections</u>: (a) N_V not an observable
- violates microcausality
(b) Needs <u>interacting</u> fields

we can try Newton-Wigner number densities, but these do not describe 'objectively localized particles

So the better approach is

(2) Vacuum fluctuations:

work with charge densities, energy densities etc

Then $[Q_V, Q_{V'}] = 0$

So microcausality satisfied

Vacuum is state of minimum total energy.

$\Rightarrow$ local observables such as Q_V fluctuate

For E.M. field, field strengths fluctuate in vacuum – explains:

spontaneous emission of radiation, Casimir effect, Lamb shift, anomalous magnetic moment of electron, ...

ALGEBRAIC QUANTUM FIELD THEORY

$$\mathcal{O} \longmapsto R(\mathcal{O})$$

└ bounded open set in space time

von Neumann algebra of observables

R acts on Hilbert space $\mathcal{H}$

$$R(\mathcal{O} + \underline{a}) = U(\underline{a}) R(\mathcal{O}) U^*(\underline{a})$$

└ representation of translation $\underline{x} \to \underline{x} + \underline{a}$

For time-like translations $U(\underline{a})$ is exponentiated to obtain a Hamiltonian operator which is non-negative.

<u>Isotony</u> For any two bounded open sets $\mathcal{O}_1, \mathcal{O}_2$, $\mathcal{O}_1 \subseteq \mathcal{O}_2 \Rightarrow R(\mathcal{O}_1) \subseteq R(\mathcal{O}_2)$

<u>Locality</u> If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like related, then $\forall A_1 \in R(\mathcal{O}_1)$, $\forall A_2 \in R(\mathcal{O}_2)$

$$[A_1, A_2] = 0$$

The Global algebra R is the smallest von N. algebra containing all the local algebras. We assume that the representation of R is irreducible and generated by the translates of $R(O)$ for any O.

The Vacuum Ω is the unique state which is invariant under all translations.

The Reeh-Schlieder Theorem

Ω is cyclic with respect to H for any $R(O)$

This just means $\{A\Omega : A \in R(O)\}$ is dense in H.

Corollary

Ω is a separating vector for $R(O)$

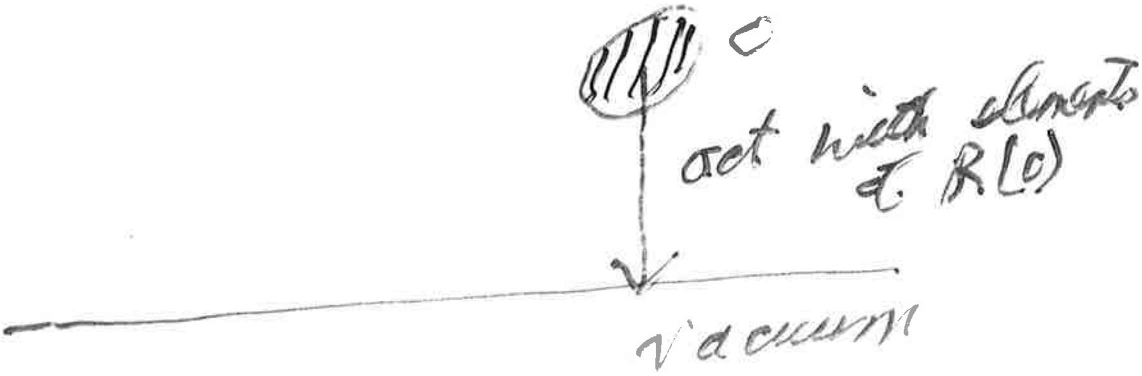
This just means
$$\forall A \in R(O), \quad A\Omega = 0 \Rightarrow A = 0.$$

I am now going to prove the R-S theorem and its corollary for a very simple analogue of a field theory, in which spacetime collapses to two points and the Von N. algebras are just the algebras of operators on a 2-dimensional Hilbert space. This is just the familiar 2 spin ½ particle system, and for the analogue of the Vacuum we shall take initially ~~[illegible]~~ the singlet

The Reeh- Schlieder Theorem

Every bounded region O of spacetime is associated with an algebra of local observables R(O).

The R-S theorem says that <u>any</u> state of the field can be generated by acting on the vacuum state with members of <u>any</u> R(O)



produces, we might expect, a local excitation

How could we generate an <u>arbitrary</u> excitation, such as

?

The Baby Reeh-Schlieder Theorem

Collapse spacetime to two points!

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Then $\Psi_{singlet}$ is cyclic for R_1 (or R_2) w.r.t. $\mathcal{H}$

$$|\Psi_{singlet}\rangle = \frac{1}{\sqrt{2}}\Big(|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$$

$$- |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle \Big)$$

Then $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_1 \in R_1$ s.t.

$$|\phi\rangle = A_1 |\Psi_{singlet}\rangle$$

Proof: By inspection.

If
$$|\phi\rangle = \alpha\, |\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$$
$$+ \beta\, |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=-1\rangle$$
$$+ \gamma\, |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$$
$$+ \delta\, |\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=+1\rangle$$

Then
$$A_1 = \alpha P_1^+ + \beta Q_1 P_1^+ + \gamma P_1^- + \delta Q_1 P_1^-$$

where $P_1^\pm$ projects the state $|\sigma_{1z}=\pm 1\rangle$ and Q_1 rotates spin 1 thro' 180°.

Similarly $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_2 \in R_2$ s.t.
$$|\phi\rangle = A_2 |\Psi_{singlet}\rangle$$

(12) (11)

Corollary

$$A_1 \, | \Psi_{singlet} \rangle = 0$$
$$\Longrightarrow A_1 = 0$$

Proof:

By the baby R–S theorem $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we can write

$$|\phi\rangle = A_2 \, |\Psi_{singlet}\rangle \text{, so}$$

$$A_1 |\phi\rangle = A_1 A_2 \, |\Psi_{singlet}\rangle$$
$$= A_2 A_1 \, |\Psi_{singlet}\rangle$$
$$= 0$$

Since $|\phi\rangle$ is any vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$, it follows that $A_1 = 0$, Q.E.D.

So $|\Psi_{singlet}\rangle$ is a cyclic vector and a separating vector for R_1 (and similarly for R_2).

We now prove a baby version of Theorem 1

Define $p = \text{Prob}^{\Psi_{singlet}} (P_1 \in R_1 = 1)$

Projector.

then $p = \| P_1 |\Psi_{singlet}\rangle \|^2$

So $p = 0 \Rightarrow P_1 |\Psi_{singlet}\rangle = 0$

$\Rightarrow P_1 = 0$ (by R-S)

$\therefore P_1 \neq 0 \Rightarrow p \neq 0$. QED.

We now turn to a baby version of Theorem 2

We want to prove.

$\forall P_2, \exists P_1$ s.t. $\langle P_1 P_2 \rangle_{\Psi_{singlet}}$

$= \langle P_1 \rangle_{\Psi_{singlet}}$

(i.e. $\text{Prob}^{\Psi_{singlet}} (P_2 = 1 / P_1 = 1) = 1$)

<u>Proof</u> Write $\Psi_{singlet} = \psi_s$

Write $|\phi\rangle = \hat{P}_2 |\psi_s\rangle / \| \hat{P}_2 |\psi_s\rangle \|$

Then by construction

$$\langle \hat{P}_2 \rangle_\phi = 1 \quad \text{———} \quad (1)$$

But, by the baby R-S theorem

$$|\phi\rangle = C_1 \, |\psi_s\rangle \quad \text{———} \quad (2)$$

where C_1 is some operator on $\mathcal{H}_1$ (extended to $\mathcal{H}_1 \otimes \mathcal{H}_2$)

Substituting (2) in (1) gives

$$\langle \psi_s | Q_1 \hat{P}_2 | \psi_s \rangle = 1 \quad --- \quad (3)$$

where $Q_1 = C_1^* C_1$, is a positive Hermitian operator on $\mathcal{H}_1$.

So we can expand

$$Q_1 = \lambda_1 \Gamma_1 + \lambda_1' \Gamma_1' \quad ---- \quad (4)$$

where λ_1, λ_1' are the non-negative real eigenvalues of Q_1, and Γ_1, Γ_1' are orthogonal projections in $\mathcal{H}_1$.

Substituting (4) in (3) yields (15)

$$w_1 \frac{\langle \Gamma_1 P_2 \rangle_{\psi_s}}{\langle \Gamma_1 \rangle_{\psi_s}} + w_2 \frac{\langle \Gamma_1' P_2 \rangle_{\psi_s}}{\langle \Gamma_1' \rangle_{\psi_s}} = 1 \quad \text{--- (5)}$$

where

$$w_1 = \lambda_1 \langle \Gamma_1 \rangle_{\psi_s}$$

$$w_2 = \lambda_1' \langle \Gamma_1' \rangle_{\psi_s}$$

But we know

$$\langle Q \rangle_{\psi_s} = \| C_1 |\psi_s\rangle \|^2$$

$$= \| |\phi\rangle \|^2 = 1$$

$$\therefore \quad w_1 + w_2 = 1 \quad \text{--- (6)}$$

with $w_1 \geq 0$, $w_2 \geq 0$.

Hence LHS (5) $\leq \max\left(\frac{\langle \Gamma_1 P_2 \rangle_{\psi_s}}{\langle \Gamma_1 \rangle_{\psi_s}}, \frac{\langle \Gamma_1' P_2 \rangle_{\psi_s}}{\langle \Gamma_1' \rangle_{\psi_s}} \right)$ ≤ 1

and (5) can only be satisfied if Γ_1 or Γ_1' (or both) satisfy the condition for Theorem 2

Q.E.D.

Now Theorems 1 and 2 are trivially true for $\Psi_{singlet}$.
Theorem 1 just says, all spin components have non-vanishing probability for results ± 1 on either particle (indeed for $\Psi_{singlet}$ all the probabilities are equal to 1/2!)
while Theorem 2 says all spin components ~~are~~ on one particle are maximally correlated with spin components on the other particle. (indeed these are just the mirror-image correlations of $\Psi_{singlet}$!)
But <u>the proofs of these well-known results for $\Psi_{singlet}$ only used the R-S theorem</u>, so they can be lifted straight back to QFT with the vacuum replacing $\Psi_{singlet}$!

In the QFT case, Theorem 2 can be formulated more accurately as:

For any two space-like separated bounded open regions O_1 and O_2 and $\forall \varepsilon > 0$, $\forall P_2 \in R(O_2)$ $\exists P_1 \in R(O_1)$ s.t.

$$\langle P_1 P_2 \rangle_\Omega > (1-\varepsilon) \langle P_1 \rangle_\Omega$$

We can also express the maximality of the correlations specified in Theorem in terms of correlation coefficients. For any two projectors P_1 and P_2 belonging to $R(O_1)$ and $R(O_2)$ respectively, we have

$$C(P_1, P_2) = \frac{\langle P_1 P_2 \rangle - \langle P_1 \rangle \cdot \langle P_2 \rangle}{[\langle P_1 \rangle \cdot (1 - \langle P_1 \rangle) \cdot \langle P_2 \rangle \cdot (1 - \langle P_2

So, for fixed $\langle P_1 \rangle$, $\langle P_2 \rangle$
the maximum value of $C(P_1, P_2)$
is given by

$$C^{max}(P_1, P_2) = \left[\frac{\langle P_1 \rangle \cdot (1 - \langle P_2 \rangle)}{\langle P_2 \rangle \cdot (1 - \langle P_1 \rangle)} \right]^{1/2} \quad \cdots (1)$$

This only attains the value 1
when $\langle P_1 \rangle = \langle P_2 \rangle$

This condition is satisfied for
$|\Psi_{singlet}\rangle$, but Theorem 2 in
no way depends on this condition.
We now want to compare (1)
with the well-known Fredenhagen bound on correlation coefficients
(Fredenhagen 1985).
This reads

$$C(P_1, P_2) \leq e^{-ml} \sqrt{(1 - \langle P_1 \rangle_\Omega) \cdot (1 - \langle P_2 \rangle_\Omega)} \quad \cdots (2)$$

where m is mass-gap, and l the minimum
Lorentz distance between O_1 and O_2

Comparing (1) and (2), consistency requires

$$\langle P_1 \rangle_\Omega \leq \frac{e^{-2m\ell} \langle P_2 \rangle_\Omega}{\left(1 - \langle P_2 \rangle_\Omega\right)^2} \quad \text{--- (3)}$$

i.e. ~~the~~ for a fixed value of $\langle P_2 \rangle_\Omega$, the maximally correlated P_1 must have a probability of occurring that falls off exponentially with the distance between O_1 and O_2.

This result shows how difficult it would be to observe the long-range correlations in the vacuum. But, of course, it does not show that they don't exist!

Turning to Theorem 3, the ambiguity referred to arises in the $\frac{1}{2}$ singlet case from the fact that the local projectors are all two-dimensional (i.e. of the form $P_1 \otimes I_2$ etc)

In QFT the technical formulation of Theorem 3 is:

$\forall P \in R(O)$, P is infinite-dimensional

<u>Proof</u> By Driessler's Theorem (1975) the von N. algebra associated with an unbounded wedge in spacetime is a type III factor.

But every bounded open region is contained in some wedge.

So, by isotony, $R(O)$ is always a sub-algebra of a type III factor.

But in a type III factor all the projectors are infinite-dimensional.

Hence all the projectors in $R(D)$ are infinite-dimensional
Q.E.D.

N.B. This result does not demonstrate that every local algebra is type III — this still remains an open question.

As a corollary of Theorem 3 we can state:
It is never a local question to ask
" Are we in the vacuum state or indeed in an N-particle state (i.e. orthogonal to the vacuum)? "

This raises the fundamental question:

What do (local) particle detectors detect?

The answer is they cannot strictly speaking be detecting particles. They detect certain types of field excitation, which for all practical purposes → FAPP may resemble particles.

But in reality (if you will excuse the phrase!) QFT is not a theory of particles, but a theory of fields and their local excitations, and that is all there is to it.

Question:

Are the vacuum fluctuations really there, if we don't observe them by some sort of measurement?

This is a generic question in the philosophy of quantum mechanics, to which we now turn.

24

Theorem 4

The Bell inequality is violated by vacuum fluctuations in spacelike separated regions

(Summers & Werner Phys. Lett. A 110, (1985) 257)

Interpretations of QM

Realism or Antirealism for possessed values
R A

Locality Principles

LOC_R : Prohibits sharp → sharp transition initiated at space-like separation

LOC_A : Prohibits unsharp → sharp transition initiated at spacelike separation.

EPR (1935) $A + LOC_A \Rightarrow R \Rightarrow \sim A$

$\therefore \quad A \Rightarrow \sim LOC_A$

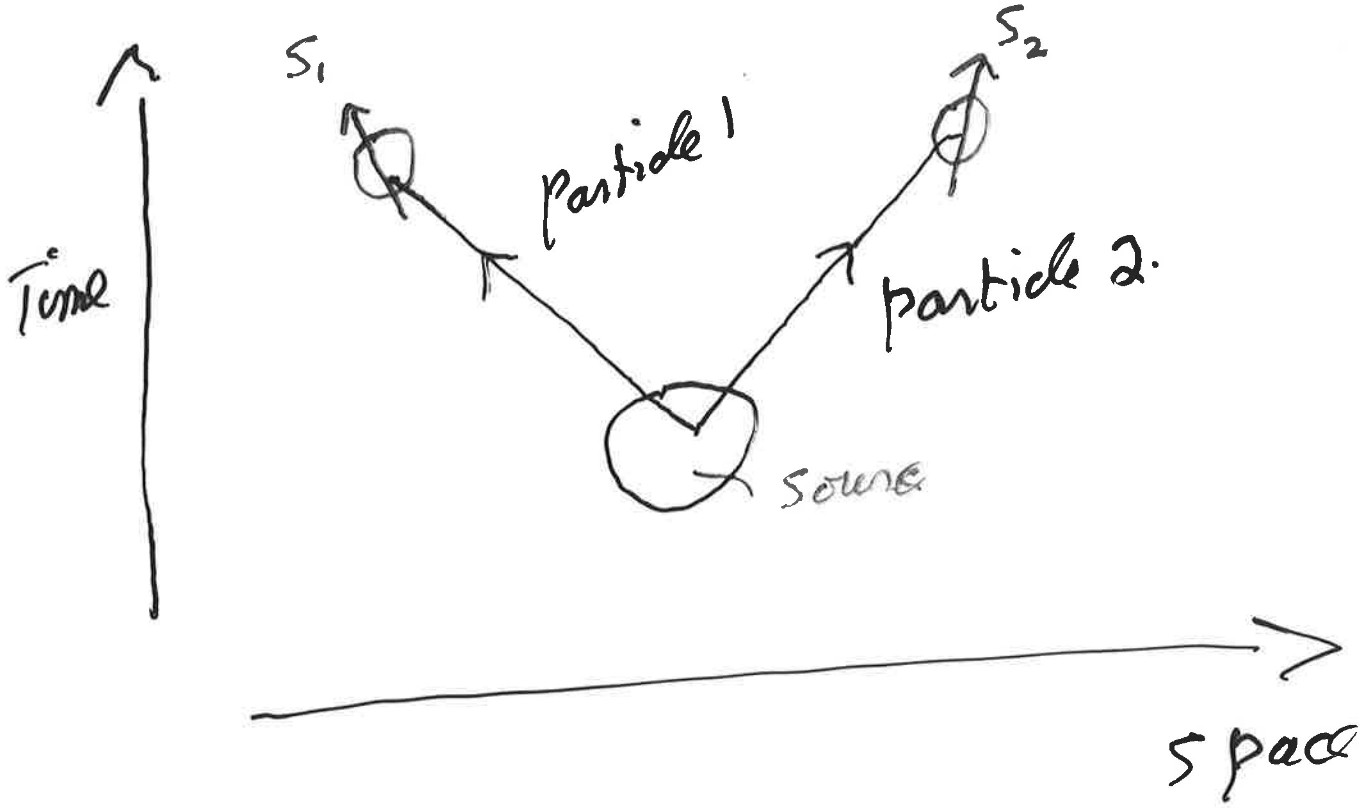
Bell (1964) $R + LOC_R \Rightarrow$ Bell Inequality (B.I)

$\therefore \quad R \Rightarrow (\sim LOC_R) \vee$ B.I.

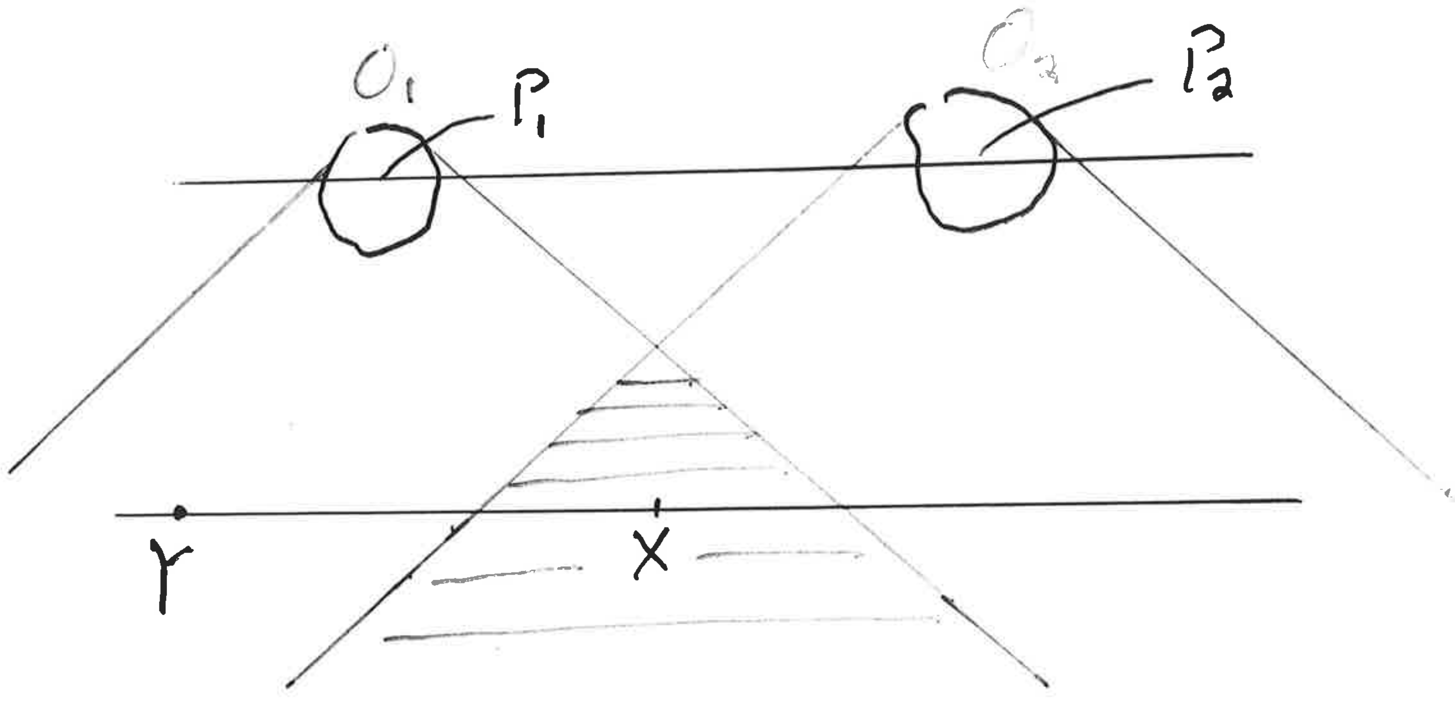
But $\sim$ B.I.

$\therefore \quad R \Rightarrow \sim LOC_R$

The Bell Experiment



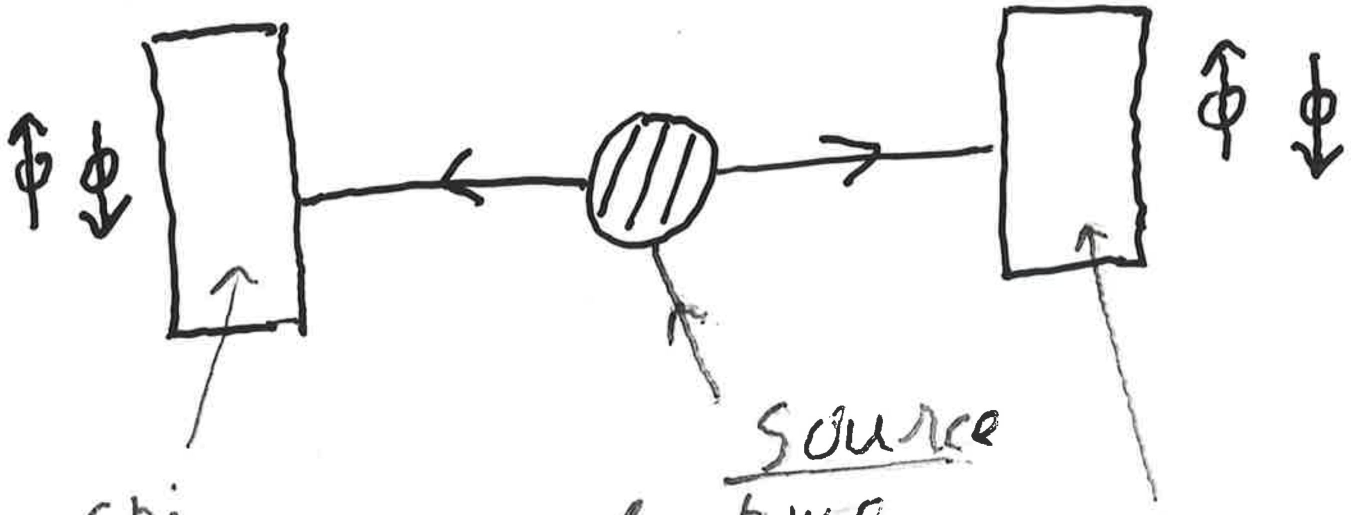
The Vacuum Version



So even if the Bell Inequality was not violated, the common cause explanation would involve an infinite regress, on pain of merely accepting the correlations at some earlier time as 'brute facts'. But if we are prepared to do this at an earlier time, why not at the later time?!

i.e. whether the B.I. is violated or not, we cannot get an acceptable local explanation of the vacuum correlations.

The EPR Experiment
(Bohm version)



Counterfactual versus Prediction in EPR

Instead of asking:

Can we predict the measurement outcome on the left, given a result O_R on the right?

we ask:

Is it the case that if we made a measurement on the left there is a definite outcome, given that a result O_R has actually occurred on the right, viz. the outcome correlated with O_R.

RELATIVISTIC EPR

(Ghirardi & Grassi
SHPMP 25 (1994) 397)

LOC_{M-O} : Prohibits measurement outcomes being affected by other measurement procedures at space-like separation.

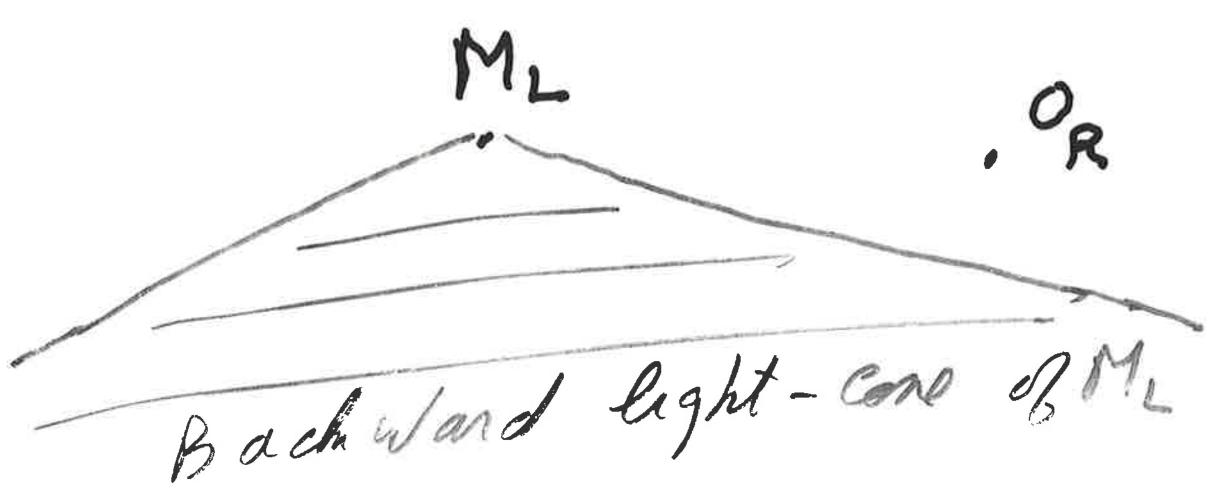
Then G & G 'prove'

$$* \quad A + LOC_A + LOC_{M-O} \Rightarrow R \Rightarrow \sim A$$

$$\therefore A \Rightarrow \sim LOC_A \vee \sim LOC_{M-O}$$

But proof of * is problematic

Assumes Determinism



Given an outcome O_R on the right, then if I made a measurement M_L on the left would O_R remain the same?

Run the world over again up to the backward light-cone of M_L - then ask, what is the outcome O_R? - it may well change on account of indeterminism, just as in the Redhead - Hellman critique of the Stapp - Eberhard proof of B.I.

Conclusion:

Assuming anti-realism of possessed values, we are unable in the vacuum state, to obtain a proof of nonlocality by the EPR-type argument. The correlations are just a 'brute fact' about the vacuum state — we cannot use the EPR argument to probe any deeper (pace Ghirardi and Grassi!).

Conclusion contd

Potential vacuum is replaced by a vacuum of potentialities

Neither Aristotle nor Einstein would have found this acceptable!

Redhead QFT

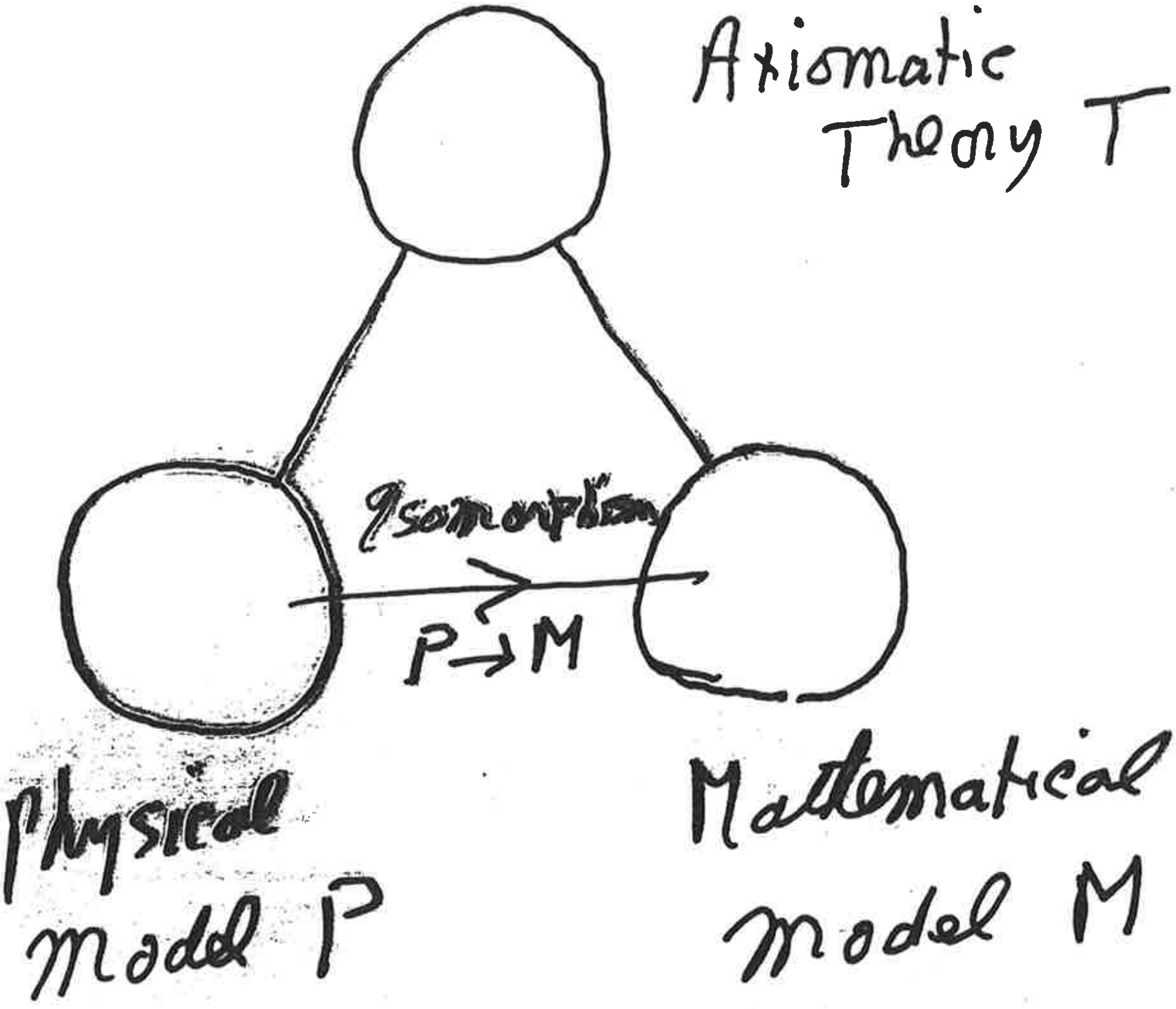
Lecture V

Gauge Theories

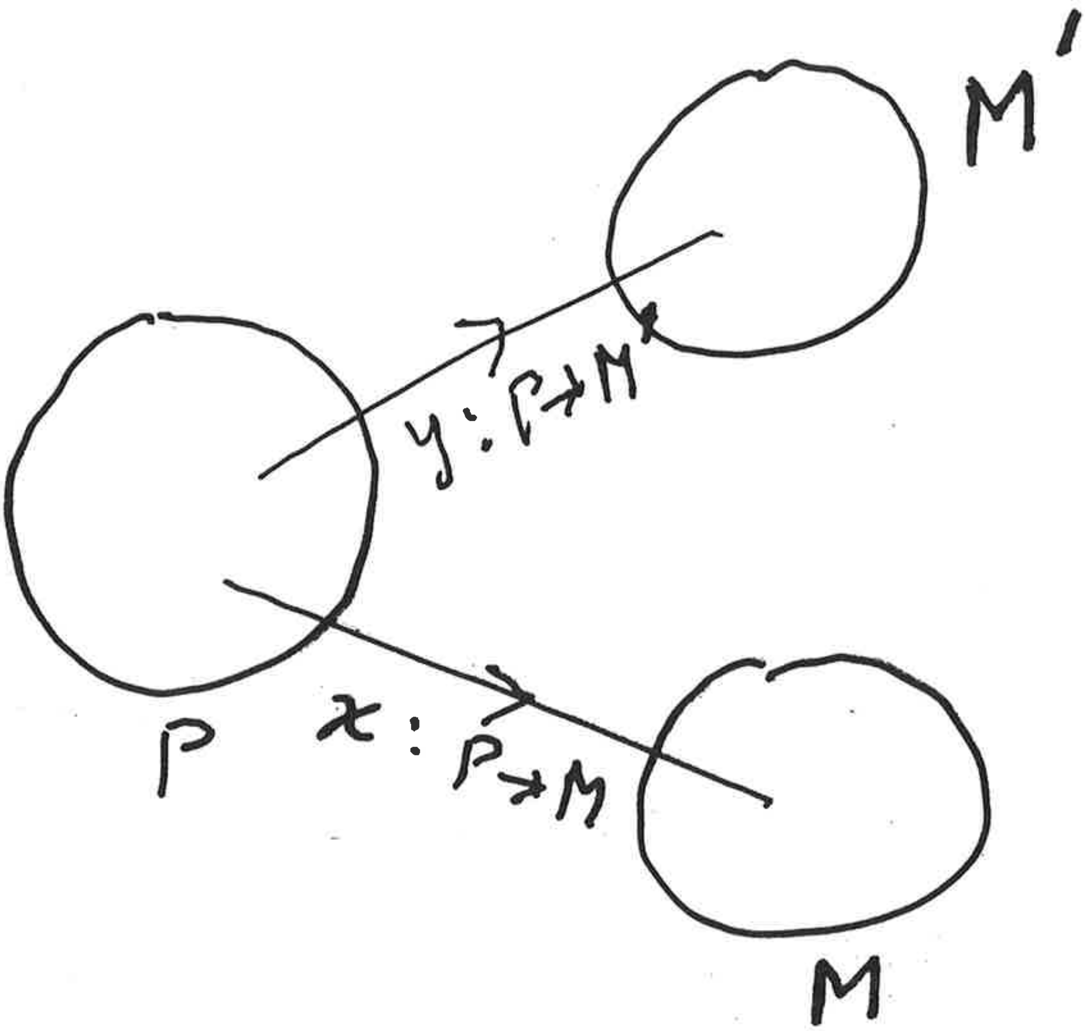
What is a gauge?

In everyday parlance 'gauge' refers to a system of measuring physical quantities, e.g. by comparing a physical magnitude with a standard or 'unit'. In this way physical magnitudes are associated with mathematical entities such as numbers.
More generally we may refer to the mathematical representation of any physical structure as a **gauge** for that structure.
Ambiguity in representation leads to the notion of **gauge transformation**

Relation of Mathematics to Physics

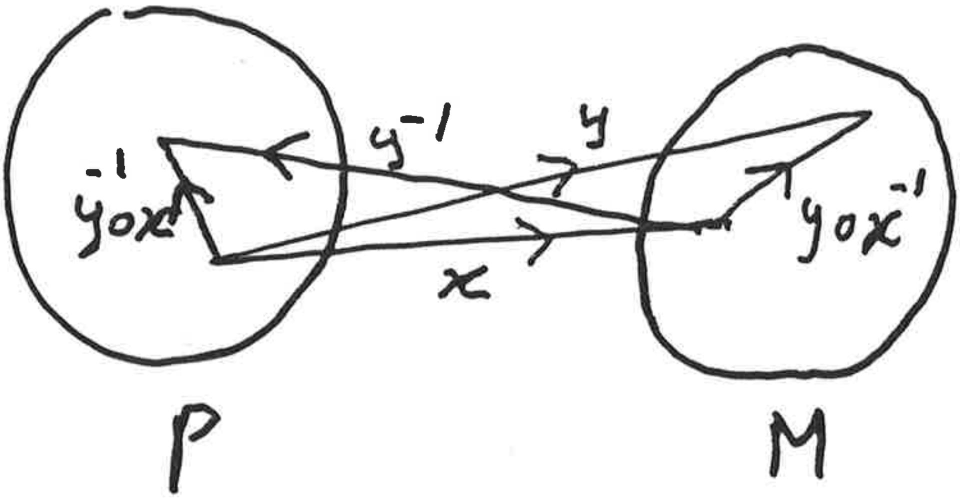


Ambiguity in Mathematical Representation



Ex Finite ordinal scales of measurement such as Moh's scale of hardness

Symmetry

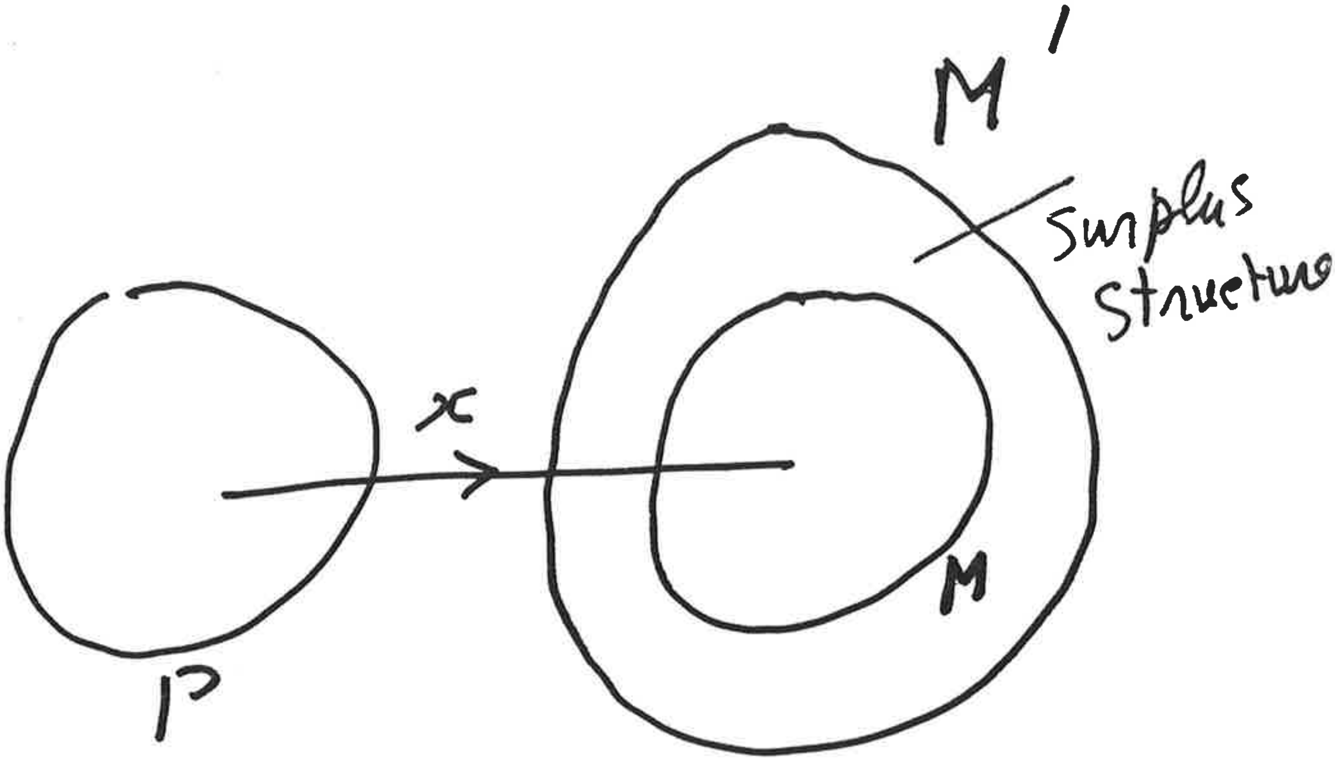


$x: P \to M$ and $y: P \to M$ are distinct isomorphisms between P and M

$y \circ x^{-1}: M \to M$ is a 'coordinate' transformation, or <u>passive</u> symmetry of P

$y^{-1} \circ x: P \to P$ is a point transformation, or <u>active</u> symmetry of P

Surplus Structure



$x : P \to M$ is an embedding of P in the larger structure M'

Ex Embedding of real line in the complex plane

Gauge Theories — Yang-Mills Type

Ex scalar electrodynamics

Classically matter field ψ is a complex-valued function on spacetime

Lagrangian is invariant under global phase transformations

$$\psi \to \psi e^{i\alpha} \qquad (1)$$

If we consider local phase transformations

$$\psi \to \psi e^{i\alpha(x)} \qquad (2)$$

where $\alpha(x)$ is an arbitrary real-valued function on spacetime

Then Lagrangian remains invariant provided we use the 'corrected' derivative $\partial_\mu - i A_\mu$, where the connection field A_μ transforms as

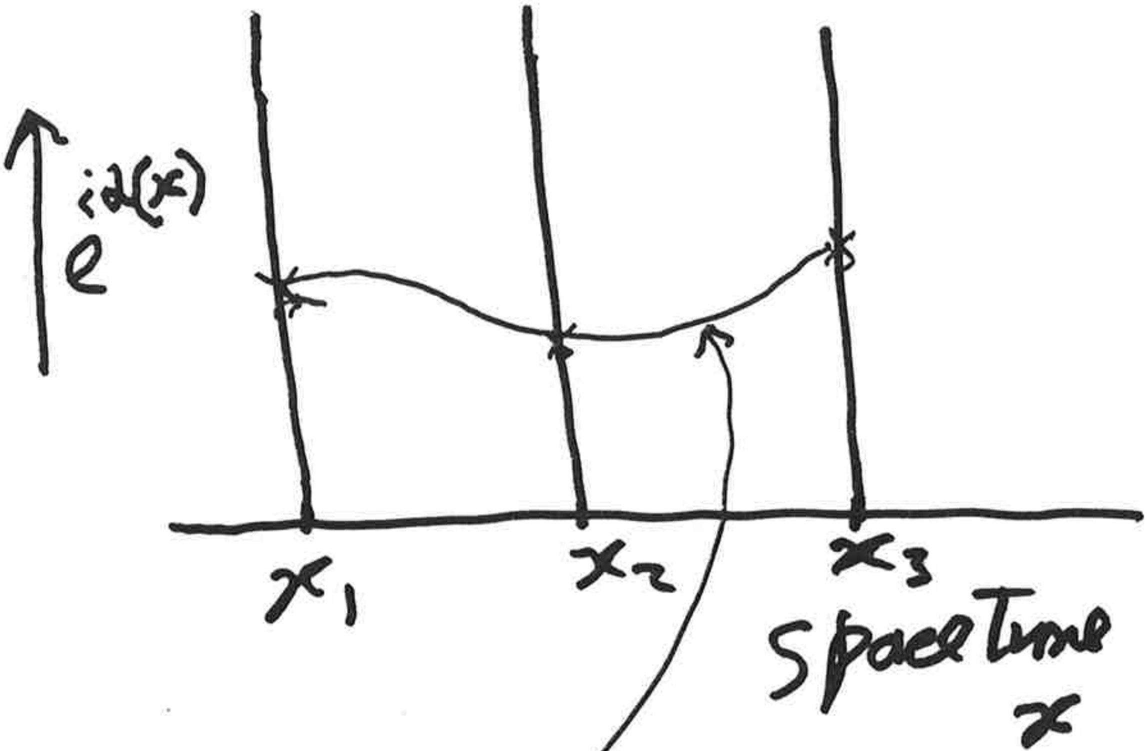
$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \quad \ldots (3)$$

A_μ can be identified, modulo the electronic charge e, with the electromagnetic <u>potential</u>.

Physically significant quantities are gauge invariant, e.g. $\psi^* \psi$ or the electromagnetic <u>field</u> $f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$.

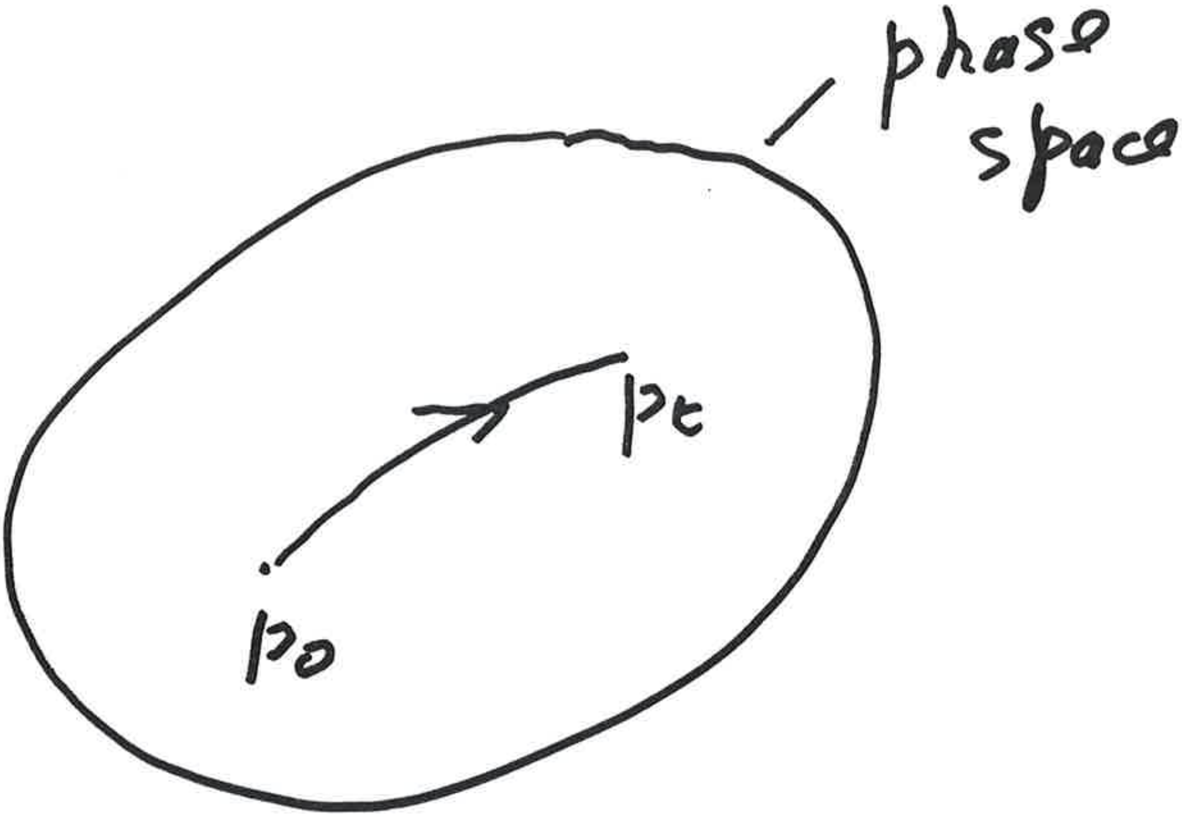
<u>N.B.</u> ψ and A_μ are elements of surplus structure, since they are not gauge invariant.

The U(1) Bundle



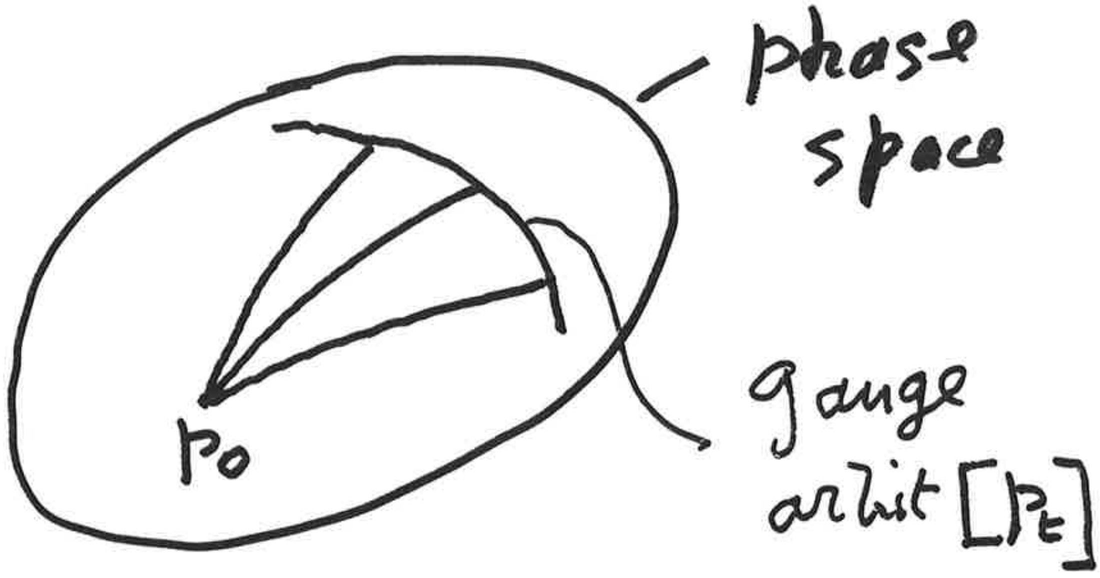
cross-section of 'constant' phase as specified by the connection field

Hamiltonian Systems



phase space is a symplectic manifold. Unique trajectory connects initial state p_0 to final state at time t, p_t.

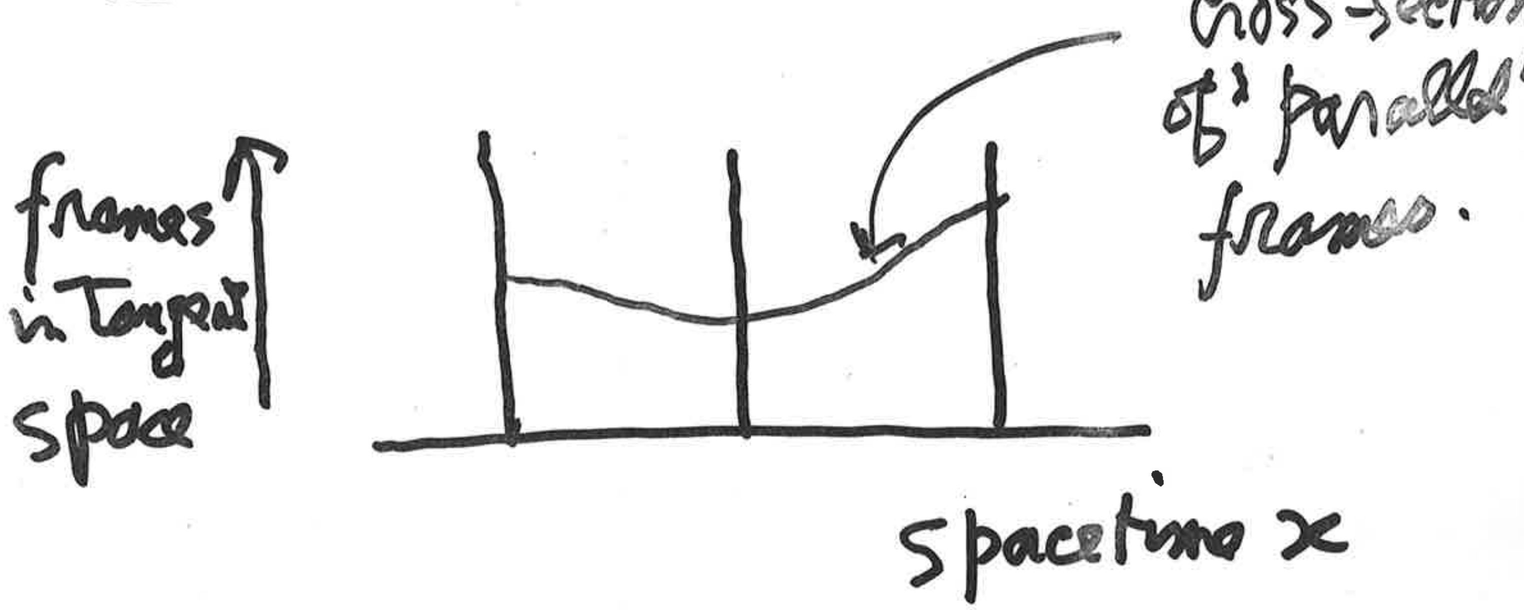
Constrained Hamiltonian Systems



phase space is a presymplectic manifold. There is no unique trajectory connecting the initial state p_0 to the final state at time t, but all the final states lie on a gauge orbit $[p_t]$ whose points are connected by gauge transformations

The Case of General Relativity

The frame bundle



Yang-Mills version

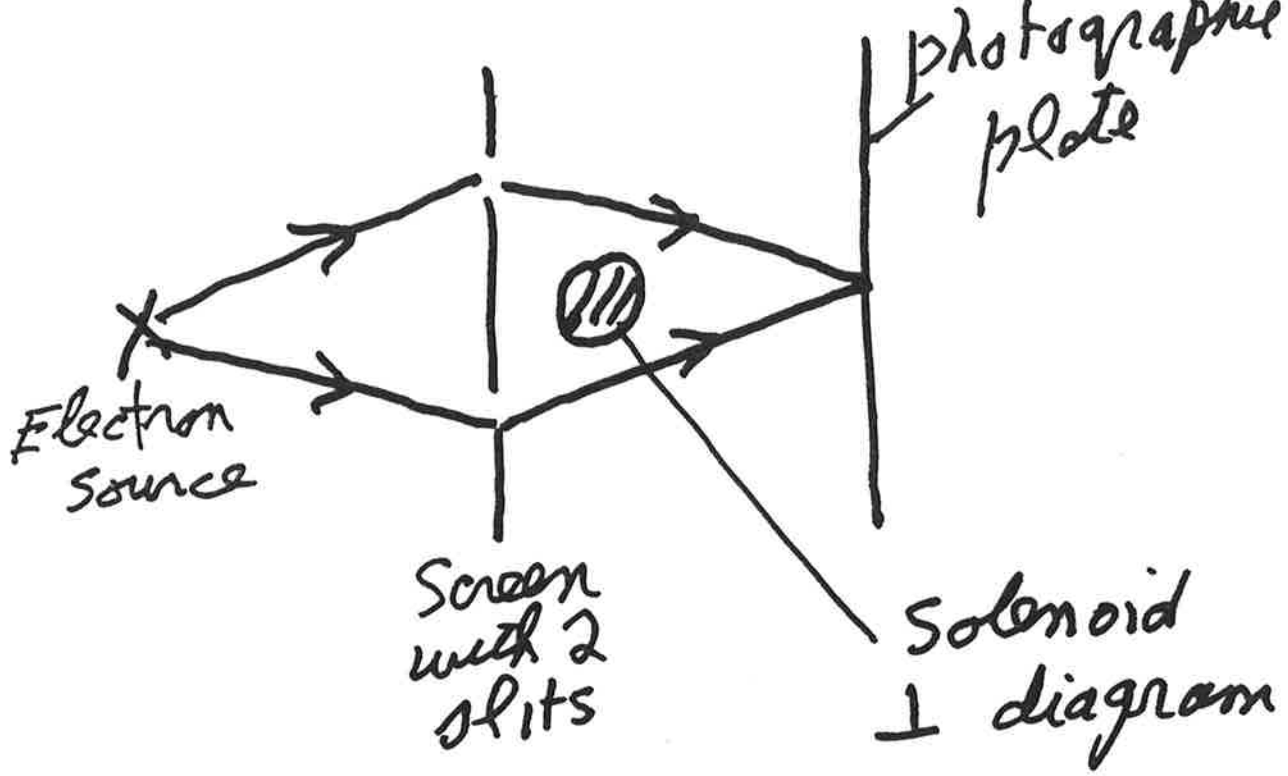
gauge group is GL(4, R) <u>acting on the fibres</u>

Constrained Hamiltonian version

Gauge group is effectively a 'manifestation' of Diff, the group of (auto) diffeomorphisms <u>acting on the base space</u>

?: Affine Bundles in extensions of GR

The Aharonov-Bohm effect



Electron 'paths' are in region of zero magnetic induction $\underline{B}$, but non-zero vector potential $\underline{A}$

The phase shift between the electrons from the 2-slits is proportional to the gauge invariant holonomy $\oint \underline{A} \cdot \underline{dl}$, which by Stokes Theorem equals the flux of $\underline{B}$ inside the solenoid

Becchi – Rouet – Stora – Tyutin (BRST) Symmetry

Ex scalar electrodynamics

We extend the ψ field and A_μ field to include other purely mathematical fields (more surplus structure!)

$$\Phi = \begin{pmatrix} \psi \\ A_\mu \\ \eta \\ \omega \\ b \end{pmatrix}$$

ψ — matter field
A_μ — gauge potential
η — ghost field
ω — anti ghost field
b — Nakanishi – Lautrup field

η and ω are scalar Grassmann fields

So $\eta^2 = \omega^2 = 0$

$$\Phi \to \Phi + \varepsilon s \Phi$$

where

$$s\Phi = \begin{pmatrix} i\eta\psi \\ \partial_\mu \eta \\ 0 \\ b \\ 0 \end{pmatrix}$$

This is a gauge transformation where the phase field is a new dynamic obj., the ghost field

ε is an infinitesimal Grassmann parameter

Notice $s^2\Phi = 0$, so s is nilpotent and behaves like an exterior derivative on the extended space of fields.

This leads to a beautiful generalized de Rham cohomology theory

Further generalizations:

① ghosts of ghosts
and ghosts of ghosts of ghosts

- - - - - !

② The Batalin – Vilkovisky antifield formalism, which introduces partners (antifields) for all the fields

<u>But</u> the antifield of a ghost is not an antighost
and the anti(antighost) is not a ghost!